Spatio-temporal representation and reasoning based on RCC-8

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1 Introduction

Qualitative representation and reasoning—as a field within AI—has been quite successful in dealing with both time and space. There exists a wide spectrum of temporal languages (see e.g. [1, 14, 32]). There is a variety of spatial formalisms (e.g. [7, 9, 21, 26]). In both cases effective reasoning procedures have been developed and implemented (e.g. [20, 18, 25, 13, 2, 29]). The next apparent and natural step would be to combine these two kinds of reasoning.

The importance of such a step for both theory and applications is beyond any doubt. And of course there have been attempts to construct spatio-temporal hybrids. For example, Clarke's [7, 8] intended interpretation of his region-based calculus was spatio-temporal. Region Connection Calculus RCC of [26] contained a function $\mathsf{space}(x, t)$ for representing the space occupied by object x at moment of time t. Muller [24] developed a first-order theory for reasoning about motion of spatial entities. However, in contrast to the 'one-dimensional' temporal and spatial cases, no effective procedures capable of reasoning about space in time have been developed.

The main aim of this paper is to introduce a hierarchy of languages intended for qualitative spatio-temporal representation and reasoning, provide these languages with topological temporal semantics, construct effective reasoning algorithms, and estimate their computational complexity.

The languages we propose are combinations of two well-known and well-understood formalisms in temporal and spatial reasoning. The spatial component is the fragment RCC-8 of RCC containing eight jointly exhaustive and pairwise disjoint base relations between spatial regions. This fragment has attracted considerable attention of the spatial reasoning community [2, 3, 19, 27, 28, 29]. First, it is sufficiently expressive for various application purposes, say in GIS. And second, RCC-8 has nice computational properties: it turns out to be decidable [2], in fact NP-complete [29]. Actually, the latter results were obtained by means of encoding $\mathsf{RCC\text{-}8}$ in the well-known propositional modal logic **S4** (whose necessity operator can be interpreted as the interior operator in topological spaces) extended with the universal modality. This makes natural the choice of the temporal component—the point based propositional temporal logic **PTL** with the binary operators 'Since' and 'Until' based on the flow of time $\langle \mathbb{N}, < \rangle$, rather than Allen's interval calculus which is spiritually closer to RCC. (We hope the examples below and the obtained results will convince the reader that these two formalisms fit together perfectly well indeed.)

PTL is one of the best known temporal logics which has found many applications in CS and AI (e.g. program verification and specification [22, 23], distributed and multi-agent systems [11], or temporal databases [6]). It is decidable and PSPACE-complete (see e.g. [14]). Thus, the problem of constructing effective spatio-temporal formalisms can be viewed as designing decidable two-dimensional modal logics one dimension of which is a topological space and another one the flow of time $\langle \mathbb{N}, < \rangle$.[1] The idea of using such kind of multi-dimensional modal logics for spatio-temporal reasoning has recently been advocated by Bennett and Cohn [4].

The computational behaviour of a combined spatio-temporal logic depends (i) on the choice of spatial and temporal operators, and (ii) on the degree of the permitted interaction between them. These two parameters give rise to a hierarchy of possible languages.

The simplest one, $\mathcal{ST}_0$, allows applications of the temporal operators Since and Until (as well as the booleans) only to $\mathsf{RCC\text{-}8}$ formulas. (Actually even this language is enough to express, for instance, the assumption that change is continuous, or the notion of conceptual neighbourhoods; see e.g. [10].) The most expressive one, $\mathcal{ST}_2^+$, makes it possible to form unions, intersections, and complementations of spatial regions, and to apply temporal operators to both formulas and region terms (for instance, $\bigcirc X$ denotes the state of region X 'tomorrow').

The natural semantics for these languages are temporal models for **PTL** each state in which is some fixed topological space. We prove that the satisfiability problem for all our languages in topological temporal models is decidable, with the computational complexity ranging from NP to EXPSPACE. We also study the structure of the simplest topological spaces that are enough to satisfy all satisfiable $\mathcal{ST}_2^+$-formulas, and consider the problem of realising spatio-temporal formulas in models based on Euclidean space.

2 Region connection calculus

Full RCC. RCC—*Region Connection Calculus*—is a first-order theory designed by Randell, Cui, and Cohn [26] for qualitative spatial representation and reasoning. The language of RCC contains only one primitive predicate $\mathsf{C}(X, Y)$ which is read 'region X *is connected* with region Y'. Starting from this, one can

[1] Actually, the obtained results can be generalized to some other flows of time, for instance $\langle \mathbb{Z}, < \rangle$ or $\langle \mathbb{R}, < \rangle$.

define other kinds of relations between spatial regions. The basic ones are:

$\mathsf{DC}(X, Y)$	'X and Y are disconnected,'
$\mathsf{EC}(X, Y)$	'X is externally connected to Y,'
$\mathsf{PO}(X, Y)$	'X partially overlaps Y,'
$\mathsf{EQ}(X, Y)$	'X is identical with Y,'
$\mathsf{TPP}(X, Y)$	'X is a tangential proper part of Y,'
$\mathsf{TPP}^{-1}(X, Y)$	'Y is a tangential proper part of X,'
$\mathsf{NTPP}(X, Y)$	'X is a nontangential proper part of Y,'
$\mathsf{NTPP}^{-1}(X, Y)$	'Y is a nontangential proper part of X.'

The intended models of RCC are topological spaces $\mathfrak{T} = \langle U, \mathbb{I} \rangle$, where U is a non-empty set, the *universe* of the space, and $\mathbb{I}$ an *interior operator* on U satisfying the usual Kuratowski axioms. Individual variables of RCC range over non-empty regular closed sets of $\mathfrak{T}$, i.e., an *assignment* in $\mathfrak{T}$ is a map $\mathfrak{a}$ associating with every variable X a set $\mathfrak{a}(X) \subseteq U$ such that $\mathfrak{a}(X) \neq \emptyset$ and $\mathfrak{a}(X) = \mathbb{C}\mathbb{I}\,\mathfrak{a}(X)$, where $\mathbb{C}$ is the closure operator on U dual to $\mathbb{I}$. The intended meaning of $\mathsf{C}(X, Y)$—'regions X and Y share at least one point'—is formalized then as follows:

$$\mathfrak{T} \models^{\mathfrak{a}} \mathsf{C}(X, Y) \text{ iff } \mathfrak{a}(X) \cap \mathfrak{a}(Y) \neq \emptyset.$$

From the computational point of view RCC is too expressive: as was observed by Gotts [17], the full first-order theory of RCC is undecidable. Fortunately, there are various decidable (and even tractable) fragments of RCC. One of the most important is known as RCC-8.

RCC-8. If we are interested only in relationships between spatial regions without taking into account their topological shape, then the eight predicates above are enough: they are jointly exhaustive and pairwise disjoint, which means that any two regions stand precisely in one of these eight relations. Moreover, according to the experiments reported by Renz and Nebel [28], the eight predicates turn out to be conceptually cognitive adequate in the sense that people indeed distinguish between those relations.

Formally, the language of RCC-8 consists of a set of individual variables, called *region variables*, the eight binary predicates DC, EC, PO, EQ, TPP, TPP^{-1}, NTPP, NTPP^{-1}, and the booleans out of which we can construct in the usual way *spatial formulas*.

The main reasoning task for RCC-8 can be formulated as follows:

- *given a finite set Σ of spatial formulas, decide whether Σ is satisfiable (or realizable) in a topological space, i.e., whether there exists a topological space $\mathfrak{T}$ and an assignments $\mathfrak{a}$ in it such that $\mathfrak{T} \models^{\mathfrak{a}} \Sigma$.*

(The predicates in Σ are replaced with their definitions via C.)

That this *satisfiability problem* is decidable was shown by Bennett [2, 3] who encoded RCC-8 in propositional intuitionistic logic and modal system **S4** using the well-known fact that both of them are complete with respect to topological spaces [30, 31]. An elementary proof of the correctness of the encoding is provided in [**?**]. To see the intuition behind this encoding, it suffices to observe that C and the eight predicates of RCC8 can be represented in the one-variable fragment of the first-order language of topological spaces. For instance,

$$\begin{array}{ll}
\mathsf{C}(X_1, X_2) & \exists x\ x \in X_1 \cap X_2, \\
\mathsf{DC}(X_1, X_2) & \neg\exists x\ x \in X_1 \cap X_2, \\
\mathsf{EC}(X_1, X_2) & \mathsf{C}(X_1, X_2) \wedge \neg\exists x\ x \in \mathbb{I}X_1 \cap \mathbb{I}X_2, \\
\mathsf{PO}(X_1, X_2) & \exists x\ x \in \mathbb{I}X_1 \cap \mathbb{I}X_2 \wedge \exists x\ x \in \mathbb{I}X_1 \cap \neg X_2 \wedge \exists x\ x \in \neg X_1 \cap \mathbb{I}X_2, \\
\mathsf{EQ}(X_1, X_2) & \forall x\ (x \in X_1 \leftrightarrow x \in X_2), \\
\mathsf{TPP}(X_1, X_2) & \forall x\ x \in \neg X_1 \cup X_2 \wedge \exists x\ x \in X_1 \cap \mathbb{C}\neg X_2 \wedge \exists x\ x \in \neg X_1 \cap X_2, \\
\mathsf{NTPP}(X_1, X_2) & \forall x\ x \in \neg X_1 \cup \mathbb{I}X_2 \wedge \exists x\ x \in \neg X_1 \cap X_2.
\end{array}$$

The requirements of non-emptiness and regularity are also expressible in this fragment:

$$\begin{array}{ll}
\mathsf{NE}(X_0) & \exists x\ x \in X_0, \\
\mathsf{Regular}(X_0) & \forall x\ x \in \mathbb{I}\neg X_0 \cup \mathbb{C}\mathbb{I}X_0.
\end{array}$$

Now, by replacing the quantifiers $\forall x$ and $\exists x$ with the universal necessity and possibility operators denoted here by $\forall$ and $\exists$, respectively, associating with every region variable X_i a propositional variable p_i and representing topological terms as the corresponding **S4**-formulas with necessity $\boldsymbol{I}$ and possibility $\boldsymbol{C}$, we obtain bimodal formulas of the form:

$$\begin{aligned}
(\mathsf{DC}(X_1, X_2))^* &= \neg\exists(p_1 \wedge p_2), \\
(\mathsf{EC}(X_1, X_2))^* &= \exists(p_1 \wedge p_2) \wedge \neg\exists(\boldsymbol{I}p_1 \wedge \boldsymbol{I}p_2), \\
(\mathsf{PO}(X_1, X_2))^* &= \exists(\boldsymbol{I}p_1 \wedge \boldsymbol{I}p_2) \wedge \exists(\boldsymbol{I}p_1 \wedge \neg p_2) \wedge \exists(\neg p_1 \wedge \boldsymbol{I}p_2), \\
(\mathsf{EQ}(X_1, X_2))^* &= \forall(p_1 \leftrightarrow p_2), \\
(\mathsf{TPP}(X_1, X_2))^* &= \forall(\neg p_1 \vee p_2) \wedge \exists(p_1 \wedge \boldsymbol{C}\neg p_2) \wedge \exists(\neg p_1 \wedge p_2), \\
(\mathsf{NTPP}(X_1, X_2))^* &= \forall(\neg p_1 \vee \boldsymbol{I}p_2) \wedge \exists(\neg p_1 \wedge p_2), \\
(\mathsf{NE}(X_0))^* &= \exists p_0, \\
(\mathsf{Regular}(X_0))^* &= \forall(\boldsymbol{I}\neg p_0 \vee \boldsymbol{C}\boldsymbol{I}p_0).
\end{aligned}$$

Given a spatial formula φ, denote by φ^* the result of replacing all occurrences of the $\mathsf{RCC\text{-}8}$ predicates $R(X_i, X_j)$ in φ by the corresponding bimodal formulas $(R(X_i, X_j))^*$. And then put

$$\varphi^\dagger = \varphi^* \wedge \bigwedge_{p \in sub\varphi^*} (\exists p \wedge \forall(\boldsymbol{I}\neg p \vee \boldsymbol{C}\boldsymbol{I}p)).$$

Having recalled the topological interpretation of **S4**, it is not hard to see that φ is satisfiable in a topological space iff $\varphi^\dagger$ is satisfiable in a Kripke model for the propositional bimodal logic $\mathbf{S4}_u$, i.e., Lewis' **S4** with universal modality. Such models have the form $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$, where $\mathfrak{F} = \langle W, R, S \rangle$ is a Kripke frame in which R is a quasi-order on W (interpreting $\boldsymbol{I}$ and $\boldsymbol{C}$), $S = W \times W$ is the universal binary relation (interpreting $\forall$ and $\exists$), and $\mathfrak{V}$ a valuation in $\mathfrak{F}$. In what follows we will omit S and write simply $\mathfrak{F} = \langle W, R \rangle$.

Moreover, using the fact that $\mathfrak{V}(p) = \boldsymbol{C}\boldsymbol{I}\mathfrak{V}(p)$ for every variable p in $\varphi^\dagger$, one can show that $\varphi^\dagger$ is satisfiable iff it is satisfied in a model based on a forest

$\mathfrak{F}$ of $\ell(\varphi^\dagger)$ trees of depth 1 the roots in which have precisely 2 successors (cf. [Renz 1998]). Here $\ell(\varphi^\dagger)$ is the *length* (say, the number of subformulas) in $\varphi^\dagger$. A proof of this fact can be found in Section 4 below.

Renz and Nebel [29] proved that the satisfiability problem for RCC-8 formulas is actually NP-complete and described a maximal tractable fragment of RCC-8.

3 Temporalizing RCC-8

Suppose now that the spatial configuration we are interested in is changing in time. Let us imagine, for instance, that the flow of time is $\langle \mathbb{N}, < \rangle$ and that we have the temporal operators $\mathcal{S}$ (Since) and $\mathcal{U}$ (Until); other standard operators like $\bigcirc$ (at the next moment), $\Box^+$ (always in the future), $\Diamond^+$ (some time in the future), etc. are defined via $\mathcal{S}$ and $\mathcal{U}$ in the usual way. We can assume also that space itself with its topology always remains the same. However, the spatial regions occupied by the objects under consideration may change with time passing by. This naïve picture is formalized by the following concept of topological temporal model.

Definition 1 (tt-model). A *topological temporal model* (or *tt-model*, for short) based on a topological space $\mathfrak{T} = \langle U, \mathbb{I} \rangle$ is a triple of the form $\mathfrak{M} = \langle \mathfrak{T}, \mathbb{N}, \mathfrak{a} \rangle$, where $\mathfrak{a}$, an *assignment* in $\mathfrak{T}$, associates with every region variable X and every moment of time $n \in \mathbb{N}$ a set $\mathfrak{a}(X, n) \subseteq U$ such that $\mathfrak{a}(X, n) = \mathbb{C}\mathbb{I}\, \mathfrak{a}(X, n)$ and $\mathfrak{a}(X, n) \neq \emptyset$. For each n, we take $\mathfrak{a}_n$ to be the function defined by $\mathfrak{a}_n(X) = \mathfrak{a}(X, n)$.

There are different ways of introducing a temporal dimension into the syntax of RCC-8.

$\mathcal{ST}_0$**.** The most obvious one RCC-8 is to allow applications of the operators $\mathcal{S}$ and $\mathcal{U}$ (along with the booleans) to spatial formulas. Denote the resulting *spatio-temporal language* by $\mathcal{ST}_0$.

Definition 2 (truth). For a tt-model $\mathfrak{M} = \langle \mathfrak{T}, \mathbb{N}, \mathfrak{a} \rangle$, an $\mathcal{ST}_0$-formula φ, and $n \in \mathbb{N}$, define the *truth-relation* $(\mathfrak{M}, n) \models \varphi$—'$\varphi$ holds in $\mathfrak{M}$ at moment n'—by induction on the construction of φ:

- if φ contains no temporal operators, then $(\mathfrak{M}, n) \models \varphi$ iff $\mathfrak{T} \models^{\mathfrak{a}_n} \varphi$;
- $(\mathfrak{M}, n) \models \varphi \mathcal{U} \psi$ iff there is $k > n$ such that $(\mathfrak{M}, k) \models \psi$ and $(\mathfrak{M}, l) \models \varphi$ for every $l \in (n, k)$, where $(n, k) = \{m : n < m < k\}$;
- $(\mathfrak{M}, n) \models \varphi \mathcal{S} \psi$ iff there is $k < n$ such that $(\mathfrak{M}, k) \models \psi$ and $(\mathfrak{M}, l) \models \varphi$ for every $l \in (k, n)$.

The interaction between time and space in $\mathcal{ST}_0$ is rather weak, and it is not hard to show that the following theorem holds.

Theorem 3. *The satisfiability problem for $\mathcal{ST}_0$-formulas in topological temporal models is PSPACE-complete.*

A proof of this and other theorems formulated in this section can be found in Section 5.

The language $\mathcal{ST}_0$ is expressive enough to capture *continuity of changes* (see e.g. [10]):

$$
\begin{aligned}
&\Box^+(\mathsf{DC}(X,Y) \to \bigcirc(\mathsf{DC}(X,Y) \lor \mathsf{EC}(X,Y))),\\
&\Box^+(\mathsf{EC}(X,Y) \to \bigcirc(\mathsf{DC}(X,Y) \lor \mathsf{EC}(X,Y) \lor \mathsf{PO}(X,Y))),\\
&\Box^+(\mathsf{PO}(X,Y) \to \bigcirc(\mathsf{EC}(X,Y) \lor \mathsf{PO}(X,Y) \lor \\
&\qquad\qquad\qquad \mathsf{TPP}(X,Y) \lor \mathsf{EQ}(X,Y) \lor \mathsf{TPP}^{-1}(X,Y))),\\
&\qquad \text{etc.}
\end{aligned}
$$

The first of these formulas, for instance, says that if two regions are disconnected at some moment, then at the next moment they either will remain disconnected or will be externally connected (but will not overlap).

However, the expressive power of $\mathcal{ST}_0$ is rather limited. In particular, we can compare regions only at one moment of time, but we are not able to connect a region as it is today with its state tomorrow to say e.g. that it is expanding or remains the same. In other words, we can express the dynamics of relations between regions, say, $\neg\Box^+\mathsf{P}(Kosovo, Yugoslavia)$ ('it is not true that Kosovo will always be part of Yugoslavia') but not the dynamics of regions themselves, e.g. that $\Box^+\mathsf{P}(EU, \bigcirc EU)$, where $\bigcirc EU$ at moment n denotes the space occupied by the EU at moment $n+1$ (so the last formula means: 'the EU will always be expanding').

This new construct may also be important to refine the continuity assumption by requiring that $\Box^+(\mathsf{EQ}(X,\bigcirc X) \lor \mathsf{O}(X,\bigcirc X))$, i.e., 'regions X and $\bigcirc X$ either coincide or overlap.'

$\mathcal{ST}_\mathbf{1}$**.** To capture this dynamics, we extend $\mathcal{ST}_0$ by allowing applications of the next-time operator $\bigcirc$ not only to formulas but also to region variables. Thus, arguments of the RCC-$\mathsf{8}$ predicates can be now region variables X prefixed by arbitrarily long sequences of $\bigcirc$, say $\bigcirc\bigcirc X$ (representing region X as it will be the day after tomorrow). Denote the resulting language by $\mathcal{ST}_1$, and let $\mathcal{ST}_1'$ be its sublanguage with only one temporal operator $\bigcirc$. Definition 2 is extended by the following clause: $\mathfrak{a}(\bigcirc X, n) = \mathfrak{a}(X, n+1)$.

Theorem 4. (i) *The satisfiability problem for $\mathcal{ST}_1$-formulas in tt-models is decidable in EXPSPACE.*

(ii) *The satisfiability problem for $\mathcal{ST}_1'$-formulas is NP-complete.*

Using $\mathcal{ST}_1$ we can express, say, that region X will always be the same (i.e., X is *rigid*): $\Box^+\mathsf{EQ}(X,\bigcirc X)$, or that it has at most two distinct states, one on even days, another on odd ones: $\Box^+\mathsf{EQ}(X,\bigcirc\bigcirc X)$. Note, by the way, that the $\mathcal{ST}_1$-formula $\Box^+\mathsf{NTPP}(X,\bigcirc X)$ is satisfiable only in models based on infinite topological spaces—unlike $\mathcal{ST}_0$- (and of course, RCC-$\mathsf{8}$) formulas for which finite topological spaces are enough.

It may appear that $\mathcal{ST}_1$ is able to compare regions only within fixed time intervals. However, using an auxiliary rigid variable X we can write, for instance, $\Box^+\mathsf{EQ}(X,\bigcirc X) \land \Diamond^+\mathsf{EQ}(X, EU) \land \mathsf{P}(Russia, X)$. This formula is satisfiable iff 'someday in the future the *present* territory of Russia will be part of the EU.' Note that the formula $\Diamond^+\mathsf{P}(Russia, EU)$ means that there will be a day when Russia—its territory on that day (say, without Chechnya but with Byelorussia)—becomes part of the EU.

Imagine now that we want to express in our spatio-temporal language that all countries in Europe will pass through the Euro-zone, but only Germany will use the euro forever. Unfortunately, we don't know which countries will be formed in Europe in the future, so we can't simply write down all formulas of the form $\Diamond^+\mathsf{P}(X, \mathit{Euro\text{-}zone})$. Hopefully, Europe is rigid, and so what we need is the possibility of constructing regions $\Diamond^+X$ and $\Box^+X$ which contain all the points that will belong to region X in the future and only common points of all future states of X, respectively. Then we can write: $\mathsf{EQ}(\mathit{Europe}, \Diamond^+\mathit{Euro\text{-}zone})$ and $\mathsf{EQ}(\mathit{Germany}, \Box^+\mathit{Euro\text{-}zone})$. The formula $\mathsf{P}(\mathit{Russia}, \Diamond^+\mathit{EU})$ says that all points of the present territory of Russia will belong to the EU in the future (but perhaps at different moments of time).

$\mathcal{ST}_\mathbf{2}$. So let us allow applications of the temporal operators $\bigcirc$, $\Box^+$, $\Diamond^+$ (and possibly their past counterparts) to region variables.

Definition 5 (region term). Every region variable is a *region term*. If t is a *region term* then so are $\bigcirc t$, $\Box^+t$, and $\Diamond^+t$.

Denote by $\mathcal{ST}_2$ the language that results from $\mathcal{ST}_0$ by allowing the use of region terms along with region variables. The intended semantics for region terms is defined as follows.

Definition 6. Let $\mathfrak{M} = \langle \mathfrak{T}, \mathbb{N}, \mathfrak{a} \rangle$ be a tt-model. Define by induction the *value* $\mathfrak{a}(t, n)$ of a region term t under $\mathfrak{a}$ at n in $\mathfrak{M}$:

- $\mathfrak{a}(\bigcirc t, n) = \mathfrak{a}(t, n+1)$,
- $\mathfrak{a}(\Box^+t, n) = \mathbb{CI} \bigcap_{k>n} \mathfrak{a}(t, k)$, and
- $\mathfrak{a}(\Diamond^+t, n) = \mathbb{CI} \bigcup_{k>n} \mathfrak{a}(t, k)$.

It is to be noted, however, that this definition uses infinite unions and intersections of regions which are rather problematic. As was observed in [26], infinite unions contradict the axioms of full RCC. The infinite intersection of a sequence of shrinking regions may result in an empty set which by definition is not a region. And finally, as we shall see below, infinite operations bring various semantical complications. To avoid this problem we can try to restrict assignments in models in such a way that infinite intersections and unions can be reduced to finite ones. There are several ways of doing this.

One idea would be to accept the *Finite Change Assumption* (FCA):

No region can change its spatial configuration infinitely often.

This means that under FCA we consider only those tt-models $\mathfrak{M} = \langle \mathfrak{T}, \mathbb{N}, \mathfrak{a} \rangle$ that satisfy the following condition: for every region term t, there is $n \in \mathbb{N}$ such that $\mathfrak{a}(t, k) = \mathfrak{a}(t, n)$ for all $k > n$. Actually, FCA can be captured by the $\mathcal{ST}_2$-formulas $\Diamond^+\Box^+\mathsf{EQ}(t, \bigcirc t)$.

Of course, FCA excludes some mathematically interesting cases. Yet, it is absolutely adequate for many applications, for example, when we are planning a job which eventually must be completed (consider a robot painting a wall). Optimists will accept FCA to describe the geography of Europe in the examples above. In temporal databases the time line is often assumed to be finite, though arbitrarily long, which corresponds to FCA.

Theorem 7. *The satisfiability problem for $\mathcal{ST}_2$-formulas in FCA-models is decidable in EXPSPACE.*

Another, more general way of reducing infinite unions and intersections to finite ones is to adopt the *Finite State Assumption* (FSA):

Every region can have only finitely many possible states (although it may change its states infinitely often).

Definition 8 (FSA-model). Say that a model $\mathfrak{M} = \langle \mathfrak{T}, \mathbb{N}, \mathfrak{a} \rangle$ satisfies FSA, or is an *FSA-model*, if for every region term t there are finitely many regular closed sets $A_1, \dots, A_m \subseteq U$ such that $\{\mathfrak{a}(t,n) : n \in \mathbb{N}\} = \{A_1, \dots, A_m\}$.

Theorem 9. (i) *The satisfiability problem for $\mathcal{ST}_2$-formulas in FSA-models is decidable in EXPSPACE.*

(ii) *An $\mathcal{ST}_2$-formula is satisfiable in an FSA-model iff it is satisfiable in an FSA-model based on a finite topological space.*

$\mathcal{ST}_{\mathbf{i}}^{+}$**.** We can make our languages $\mathcal{ST}_i$, for $i = 0, 1, 2$, even more expressive by allowing applications of the boolean operations to region terms. Their semantical meaning is defined as follows:

- $\mathfrak{a}(t \vee t', n) = \mathfrak{a}(t,n) \cup \mathfrak{a}(t',n)$,
- $\mathfrak{a}(t \wedge t', n) = \mathbb{CI}(\mathfrak{a}(t,n) \cap \mathfrak{a}(t',n))$,
- $\mathfrak{a}(\neg t, n) = \mathbb{CI}(U - \mathfrak{a}(t,n))$.

So we can write, e.g.

$$\mathsf{EQ}(UK, Great\_Britain \vee Northern\_Ireland).$$

Let $\mathcal{ST}_i^{+}$ be the resulting family of languages. It turns out that all theorems above remain valid after replacing $\mathcal{ST}_i$ with $\mathcal{ST}_i^{+}$.

4 Modal encoding of $\mathcal{ST}_2^{+}$.

The decidability and complexity results formulated above are proved in Section 5 by means of embedding $\mathcal{ST}_2^{+}$ into a temporalized version of the propositional modal logic **S4** extended with the universal modality [16] and then using the method of quasimodels developed in [33]. Denote by $\mathcal{ML}$ the propositional modal language whose connectives are the booleans, the necessity and possibility operators $\boldsymbol{I}$ and $\boldsymbol{C}$ of **S4**, the universal necessity and possibility operators $\forall$ and $\exists$, and the temporal operators $\mathcal{S}$ and $\mathcal{U}$. What is the intended semantics of $\mathcal{ML}$? When encoding pure RCC-8 in **S4** with $\forall$, we can use both (more general) topological models of **S4** and (more transparent) Kripke models (which is explained, for instance, by the fact that **S4** has the finite model property). The addition of the temporal component makes the situation more complicated. For we can interpret $\mathcal{ML}$-formulas in structures of two kinds that are not equivalent with respect to these formulas.

Definition 10 (Kripke model). A *Kripke $\mathcal{ML}$-model* is a triple of the form $\mathfrak{K} = \langle \mathfrak{F}, \mathbb{N}, \mathfrak{V} \rangle$ where $\mathfrak{F} = \langle W, R \rangle$ is a quasi-order (a frame for **S4**) and $\mathfrak{V}$, a *valuation*, is a map associating with every propositional variable p and every $n \in \mathbb{N}$ a subset $\mathfrak{V}(p, n) \subseteq W$. The truth-relation $(u, n) \models_{\mathfrak{K}} \varphi$ in $\mathfrak{K}$ is defined as follows:

- $(u, n) \models_{\mathfrak{K}} p$ iff $u \in \mathfrak{V}(p, n)$,
- $(u, n) \models_{\mathfrak{K}} \forall\psi$ iff $(v, n) \models_{\mathfrak{K}} \psi$ for all $v \in W$,
- $(u, n) \models_{\mathfrak{K}} \boldsymbol{I}\psi$ iff $(v, n) \models_{\mathfrak{K}} \psi$ for all $v \in W$ such that uRv,
- $(u, n) \models_{\mathfrak{K}} \psi\mathcal{U}\chi$ iff there is $k > n$ such that $(u, k) \models_{\mathfrak{K}} \chi$ and $(u, m) \models_{\mathfrak{K}} \psi$ for all $m \in (n, k)$,
- $(u, n) \models_{\mathfrak{K}} \psi\mathcal{S}\chi$ iff there is $k < n$ such that $(u, k) \models_{\mathfrak{K}} \chi$ and $(u, m) \models_{\mathfrak{K}} \psi$ for all $m \in (k, n)$,

plus the standard clauses for the booleans. An $\mathcal{ML}$-formula φ is *satisfied* in $\mathfrak{K}$ if $(u, n) \models_{\mathfrak{K}} \varphi$ for some $u \in W$ and $n \in \mathbb{N}$.

Definition 11 (topological model). A *topological model* for $\mathcal{ML}$ is the structure $\mathfrak{N} = \langle \mathfrak{T}, \mathbb{N}, \mathfrak{U} \rangle$ in which $\mathfrak{T} = \langle U, \mathbb{I} \rangle$ is a topological space and $\mathfrak{U}$, a *valuation*, is a map associating with every propositional variable p and every $n \in \mathbb{N}$ a set $\mathfrak{U}(p, n) \subseteq U$. $\mathfrak{U}$ is then extended to arbitrary $\mathcal{ML}$-formulas in the following way:

- $\mathfrak{U}(\psi \wedge \chi, n) = \mathfrak{U}(\psi, n) \cap \mathfrak{U}(\chi, n)$,
- $\mathfrak{U}(\neg\psi, n) = U - \mathfrak{U}(\psi, n)$,
- $\mathfrak{U}(\forall\psi, n) = U$ if $\mathfrak{U}(\psi, n) = U$, and $\mathfrak{U}(\forall\psi, n) = \emptyset$ otherwise,
- $\mathfrak{U}(\boldsymbol{I}\psi, n) = \mathbb{I}\mathfrak{U}(\psi, n)$,
- $x \in \mathfrak{U}(\psi\mathcal{U}\chi, n)$ iff there is $k > n$ such that $x \in \mathfrak{U}(\chi, k)$ and $x \in \mathfrak{U}(\psi, m)$ for all $m \in (n, k)$,
- $x \in \mathfrak{U}(\psi\mathcal{S}\chi, n)$ iff there is $k < n$ such that $x \in \mathfrak{U}(\chi, k)$ and $x \in \mathfrak{U}(\psi, m)$ for all $m \in (k, n)$.

In particular, $\mathfrak{U}(\Diamond^+\psi, n) = \bigcup_{k>n} \mathfrak{U}(\psi, k)$ and $\mathfrak{U}(\Box^+\psi, n) = \bigcap_{k>n} \mathfrak{U}(\psi, k)$. An $\mathcal{ML}$-formula φ is *satisfied* in $\mathfrak{N}$ if $\mathfrak{U}(\varphi, n) \neq \emptyset$ for some $n \in \mathbb{N}$.

The sets of $\mathcal{ML}$-formulas satisfiable in Kripke models and topological models turn out to be different. Of course, every Kripke model (based on $\langle W, R \rangle$) is equivalent to some topological model (with $\langle 2^W, \mathbb{I} \rangle$ as the underlying topological space, where $\mathbb{I}X$ is the maximal R-closed subset of X); see e.g. [5]. But the converse does not hold.

Proposition 12. *The formula $\Diamond^+\boldsymbol{C}p \leftrightarrow \boldsymbol{C}\Diamond^+p$ is valid in every Kripke $\mathcal{ML}$-model but not in every topological $\mathcal{ML}$-model.*

Proof The former claim is clear. To see that there is a topological $\mathcal{ML}$-model in which the formula is refuted, it suffices to take $\mathfrak{T} = \langle \mathbb{R}, \mathbb{I} \rangle$ with the standard interior operator on the real line, select a sequence X_n of closed sets such that $\bigcup_{n \in \mathbb{N}} X_n$ is not closed, and put $\mathfrak{U}(p, n) = X_n$. ❑

Fortunately, the two types of models are equivalent with respect to *modal translations* of $\mathcal{ST}_2^+$-formulas under the finite state assumption FSA. The modal translation † from $\mathcal{ST}_2^+$ into $\mathcal{ML}$ is defined as follows.

Definition 13 (modal translation). For a region term t, define an $\mathcal{ML}$-formula t^* by taking

$$
\begin{array}{ll}
X_i^* = p_i, \ X_i \text{ a region variable}, & (\bigcirc t)^* = \boldsymbol{CI} \bigcirc t^*, \\
(\Diamond^+ t)^* = \boldsymbol{CI}\Diamond^+ t^*, & (\Box^+ t)^* = \boldsymbol{CI}\Box^+ t^*, \\
(t_1 \vee t_2)^* = \boldsymbol{CI}(t_1^* \vee t_2^*), & (t_1 \wedge t_2)^* = \boldsymbol{CI}(t_1^* \wedge t_2^*), \\
(\neg t)^* = \boldsymbol{CI}\neg t^*. &
\end{array}
$$

For atomic $\mathcal{ST}_2^+$-formulas, let

$$
\begin{array}{l}
(\mathsf{DC}(t_1,t_2))^* = \neg\exists(t_1^* \wedge t_2^*), \\
(\mathsf{EC}(t_1,t_2))^* = \exists(t_1^* \wedge t_2^*) \wedge \neg\exists(\boldsymbol{I}t_1^* \wedge \boldsymbol{I}t_2^*), \\
(\mathsf{PO}(t_1,t_2))^* = \exists(\boldsymbol{I}t_1^* \wedge \boldsymbol{I}t_2^*) \wedge \exists(\boldsymbol{I}t_1^* \wedge \neg t_2^*) \wedge \exists(\neg t_1^* \wedge \boldsymbol{I}t_2^*), \\
(\mathsf{EQ}(t_1,t_2))^* = \forall(t_1^* \leftrightarrow t_2^*), \\
(\mathsf{TPP}(t_1,t_2))^* = \forall(\neg t_1^* \vee t_2^*) \wedge \exists(t_1^* \wedge \boldsymbol{C}\neg t_2^*) \wedge \exists(\neg t_1^* \wedge t_2^*), \\
(\mathsf{TPP}^{-1}(t_1,t_2))^* = (\mathsf{TPP}(t_2,t_1))^*, \\
(\mathsf{NTPP}(t_1,t_2))^* = \forall(\neg t_1^* \vee \boldsymbol{I}t_2^*) \wedge \exists(\neg t_1^* \wedge t_2^*), \\
(\mathsf{NTPP}^{-1}(t_1,t_2))^* = (\mathsf{NTPP}(t_2,t_1))^*, \\
(\mathsf{NE}(t))^* = \exists t^*, \\
(\mathsf{Regular}(t))^* = \forall(\boldsymbol{I}\neg t^* \vee \boldsymbol{CI}t^*).
\end{array}
$$

Suppose now that φ is an arbitrary $\mathcal{ST}_2^+$-formula. Denote by φ^* the result of replacing all occurrences of RCC-8 predicates $R(t_1,t_2)$ in φ by $(R(t_1,t_2))^*$ and then put

$$\varphi^\dagger = \varphi^* \wedge \bigwedge_{t \in term\varphi} (\mathsf{NE}(t))^* \wedge (\mathsf{Regular}(t))^*,$$

where $term\varphi$ is the set of all region terms occurring in φ. The $\mathcal{ML}$-formula $\varphi^\dagger$ is called the *modal translation* of φ.

Say that a topological $\mathcal{ML}$-model $\langle \mathfrak{T}, \mathbb{N}, \mathfrak{U} \rangle$ (or a Kripke $\mathcal{ML}$-model $\langle \mathfrak{F}, \mathbb{N}, \mathfrak{U} \rangle$) *satisfies FSA* for an $\mathcal{ML}$-formula φ if for every variable p occurring in φ and every $n \in \mathbb{N}$ there exist finitely many sets $A_1, \dots, A_k$ such that

$$\{\mathfrak{U}(p,m) : m > n\} = \{A_1, \dots, A_k\}.$$

It is easy to show by induction that in this case we also have that for every $\mathcal{ML}$-formula ψ built up from variables in φ and every $n \in \mathbb{N}$ there are sets $B_1, \dots, B_l$ such that

$$\{\mathfrak{U}(\psi,m) : m > n\} = \{B_1, \dots, B_l\}.$$

By a straightforward induction one can prove the following:

Theorem 14. *An $\mathcal{ST}_2^+$-formula is satisfiable in a tt-model (with FSA) iff its modal translation is satisfiable in a topological $\mathcal{ML}$-model (with FSA).*

The modal translations of $\mathcal{ST}_2^+$-formulas form a rather special fragment of the modal language $\mathcal{ML}$. Renz [27] showed that an RCC-8 formula φ is satisfiable iff $\varphi^\dagger$ is satisfiable in a Kripke model based on an **S4**-frame of depth ≤ 1 and width ≤ 2 (which means that it contains no chains of more than 2 distinct points, and no point has more than 2 distinct successors). It turns out that this result can be generalized to $\mathcal{ST}_2^+$-formulas.

Definition 15 ($\boldsymbol{CI}$-term). We will be distinguishing between four types of $\boldsymbol{CI}$-terms:

- an $\mathcal{ML}$-formula is called a *$\boldsymbol{CI}$-term* if it can be obtained from a propositional temporal formula χ with operators $\bigcirc$, $\Diamond^+$, $\Box^+$ by prefixing $\boldsymbol{CI}$ to every subformula of χ;
- an $\mathcal{ML}$-formula is called a *$\boldsymbol{CI}_\bigcirc$-term* if it can be obtained from a propositional temporal formula χ with only one temporal operator $\bigcirc$ by prefixing $\boldsymbol{CI}$ to every subformula of χ;
- an $\mathcal{ML}$-formula φ is a *general $\boldsymbol{CI}$-term* (*general $\boldsymbol{CI}_\bigcirc$-term*) if it is a $\boldsymbol{CI}$-term (respectively, $\boldsymbol{CI}_\bigcirc$-term) prefixed by a string of $\neg$, $\boldsymbol{I}$, and $\boldsymbol{C}$.

Remark 16. Since $\boldsymbol{CI}$-terms begin with $\boldsymbol{CI}$, every general $\boldsymbol{CI}$-term is equivalent in topological models to a term of the form χ, $\neg\chi$, $\boldsymbol{I}\chi$, $\neg\boldsymbol{I}\chi$, or $\boldsymbol{I}\neg\chi$, with χ being a $\boldsymbol{CI}$-term.

Definition 17 ($\boldsymbol{CI}$-formula). By a *$\boldsymbol{CI}$-formula* we mean an $\mathcal{ML}$-formula composed from formulas of the form $\exists\psi$ and $\forall\psi$, where ψ is a boolean combination of general $\boldsymbol{CI}$-terms, using the temporal operators and the booleans. A *$\boldsymbol{CI}_\bigcirc$-formula* is an $\mathcal{ML}$-formula composed from formulas of the form $\exists\psi$ and $\forall\psi$, where ψ is a boolean combination of general $\boldsymbol{CI}_\bigcirc$-terms, using arbitrary temporal operators and the booleans.

It easily follows from the given definitions that:

- the modal translation of every $\mathcal{ST}_1$-formula is equivalent (in topological $\mathcal{ML}$-models) to a $\boldsymbol{CI}_\bigcirc$-formula.
- the modal translation of every $\mathcal{ST}_2^+$-formula is equivalent (in topological $\mathcal{ML}$-models) to a $\boldsymbol{CI}$-formula.

We now show that all $\boldsymbol{CI}$-formulas satisfiable in topological $\mathcal{ML}$-models with FSA and all satisfiable $\boldsymbol{CI}_\bigcirc$-formulas can be satisfied in Kripke $\mathcal{ML}$-models of a rather simple form.

The following lemma is actually based on the Stone–Jónsson–Tarski representation of topological boolean algebras, in particular, topological spaces, in the form of general frames (for definitions consult [15] or [5]).

Lemma 18. (i) *If a $\boldsymbol{CI}$-formula φ is satisfied in a topological $\mathcal{ML}$-model with FSA, then φ is satisfied in a Kripke $\mathcal{ML}$-model with the FSA.*

(ii) *If a $\boldsymbol{CI}_\bigcirc$-formula φ is satisfied in a topological $\mathcal{ML}$-model, then φ is satisfied in a Kripke $\mathcal{ML}$-model as well.*

Moreover, in both cases we can choose a Kripke model $\mathfrak{K} = \langle\mathfrak{F}, \mathbb{N}, \mathfrak{V}\rangle$ satisfying φ and $\mathfrak{F} = \langle W, R\rangle$ in such a way that every set of the form

$$\{u \in W : vRu \text{ and } (u,n) \models_{\mathfrak{K}} \psi\},$$

where $v \in W$, $n \in \mathbb{N}$, and ψ is a $\boldsymbol{CI}$-term, contains an R-maximal point.

Proof Suppose φ is satisfied in a topological model $\mathfrak{N} = \langle \mathfrak{T}, \mathbb{N}, \mathfrak{U} \rangle$ based on a topological space $\mathfrak{T} = \langle U, \mathbb{I} \rangle$. Denote by W the set of ultrafilters in 2^U and for any two ultrafilters $v_1, v_2 \in W$ put

$$v_1 R v_2 \text{ iff } \forall V \subseteq U \ (\mathbb{I}V \in v_1 \to V \in v_2).$$

It can be shown that R is a quasi-order on W (see [5]). Put $\mathfrak{F} = \langle W, R \rangle$ and define a Kripke model $\mathfrak{K} = \langle \mathfrak{F}, \mathbb{N}, \mathfrak{V} \rangle$ by taking

$$\mathfrak{V}(p, n) = \{v \in W : \mathfrak{U}(p, n) \in v\}.$$

Now, by induction on the construction of $\psi \in sub\varphi$ we show that for all $v \in W$ and $n \in \mathbb{N}$, we have

$$(v, n) \models_{\mathfrak{K}} \psi \quad \text{iff} \quad \mathfrak{U}(\psi, n) \in v.$$

The basis of induction follows from definition, and the cases of the booleans and the modal operators of **S4** are standard (see e.g. [5]).

Suppose that $(v, n) \models_{\mathfrak{K}} \forall\psi$. Then $\forall u \in W \ (u, n) \models_{\mathfrak{K}} \psi$ and so, by IH, $\forall u \in W \ \mathfrak{U}(\psi, n) \in u$, which means that $\mathfrak{U}(\psi, n) = U$, whence $\mathfrak{U}(\forall\psi, n) = U \in v$. Conversely, if $\mathfrak{U}(\forall\psi, n) \in v$ then $\mathfrak{U}(\forall\psi, n) \neq \emptyset$ and so $\mathfrak{U}(\forall\psi, n) = U$. It follows that $\mathfrak{U}(\psi, n) = U$, i.e., $\forall u \in W \ \mathfrak{U}(\psi, n) \in u$, from which by IH $(v, n) \models_{\mathfrak{K}} \forall\psi$.

The case of $\bigcirc\psi$ is easy. We have $(v, n) \models_{\mathfrak{K}} \bigcirc\psi$ iff $(v, n+1) \models_{\mathfrak{K}} \psi$ iff, by IH, $\mathfrak{U}(\psi, n+1) = \mathfrak{U}(\bigcirc\psi, n) \in v$.

Now assume that $(v, n) \models_{\mathfrak{K}} \Diamond^+\psi$. Then $\exists m > n \ (v, m) \models_{\mathfrak{K}} \psi$ and so, by IH, $\mathfrak{U}(\psi, m) \in v$. As $\mathfrak{U}(\Diamond^+\psi, n) = \bigcup_{k>n} \mathfrak{U}(\psi, k)$, we have $\mathfrak{U}(\psi, m) \subseteq \mathfrak{U}(\Diamond^+\psi, n)$, from which $\mathfrak{U}(\Diamond^+\psi, n) \in v$.

Conversely, let $\mathfrak{U}(\Diamond^+\psi, n) = \bigcup_{k>n} \mathfrak{U}(\psi, k) \in v$. By FSA, we then have

$$\mathfrak{U}(\Diamond^+\psi, n) = \mathfrak{U}(\psi, k_1) \cup \dots \cup \mathfrak{U}(\psi, k_l) \in v$$

for some $k_1, \dots, k_l > n$. Since v is a prime filter, $\mathfrak{U}(\psi, k_m) \in v$ for some m, from which by IH, $(v, k_m) \models_{\mathfrak{K}} \psi$ and so $(v, n) \models_{\mathfrak{K}} \Diamond^+\psi$.

The case of $\Box^+\psi$ is treated similarly. Note that we use FSA only when φ contains a $\boldsymbol{CI}$-term with $\Diamond^+$ or $\Box^+$.

The existence of R-maximal points follows from [12] (see Theorem 10.36 in [5]). ❑

A quasi-order $\langle V, S \rangle$ is said to be of *depth* ≤ 1 if V can be represented as the disjoint union of two sets, V_1 and V_0, in such a way that S is the reflexive closure of a subset of $V_1 \times V_0$. The points in V_i are said to be of *depth* i.

Lemma 19. *Suppose that a $\boldsymbol{CI}$-formula φ is satisfied in a Kripke $\mathcal{ML}$-model $\mathfrak{M} = \langle \mathfrak{G}, \mathbb{N}, \mathfrak{U} \rangle$ with $\mathfrak{G} = \langle W, R \rangle$. Suppose also that for any $w \in W$, $n \in \mathbb{N}$, and any $\boldsymbol{CI}$-formula ψ the set $\{u \in W : wRu$ and $(u, n) \models_{\mathfrak{M}} \psi\}$ contains an R-maximal point. Then φ is satisfied in a Kripke model $\mathfrak{K} = \langle \mathfrak{F}, \mathbb{N}, \mathfrak{V} \rangle$ in which $\mathfrak{F}$ is a quasi-order of depth ≤ 1. If $\mathfrak{M}$ satisfies FSA, then $\mathfrak{K}$ satisfies FSA as well.*

Proof Define $\mathfrak{F} = \langle V, S \rangle$ by taking $V = V_0 \cup V_1$, where

$$V_0 = \{x \in W : \neg\exists y \ (xRy \wedge \neg yRx)\}, \quad V_1 = W - V_0,$$

and taking S to be the reflexive closure of $R \cap (V_1 \times V_0)$. In other words, $\mathfrak{F}$ keeps the same set of worlds as $\mathfrak{G}$, but only those arrows from the latter that lead to points in final clusters (arrows within these clusters are omitted). By the condition of the lemma, $V_0 \neq \emptyset$ (take $\psi = \top$). Finally, we put $\mathfrak{V} = \mathfrak{U}$ and $\mathfrak{K} = \langle \mathfrak{F}, \mathbb{N}, \mathfrak{V} \rangle$.

First we show that for every $\boldsymbol{CI}$-term ψ, every $n \in \mathbb{N}$, and every $u \in V$,

$$(u,n) \models_{\mathfrak{K}} \psi \quad \text{iff} \quad (u,n) \models_{\mathfrak{M}} \psi. \tag{1}$$

The proof is by induction. The basis of induction follows from the definition. Now recall that every $\boldsymbol{CI}$-term begins with $\boldsymbol{CI}$. And for such formulas we clearly have:

$$(u,n) \models_{\mathfrak{M}} \boldsymbol{CI}\chi \text{ iff } \exists v \in W\ (uRv \ \&\ \forall w \in W\ (vRw \to (w,n) \models_{\mathfrak{M}} \chi.$$

Therefore, $(u,n) \models_{\mathfrak{M}} \boldsymbol{CI}\chi$ iff there is a point $v \in V_0$ such that uSv and $(v,n) \models_{\mathfrak{M}} \chi$. Suppose, for instance, that $\chi = \bigcirc\psi$. Then $(v,n) \models_{\mathfrak{M}} \bigcirc\psi$ iff $(v,n+1) \models_{\mathfrak{M}} \psi$ iff, by IH, $(v,n+1) \models_{\mathfrak{K}} \psi$ iff $(v,n) \models_{\mathfrak{K}} \chi$ iff $(u,n) \models_{\mathfrak{K}} \boldsymbol{CI}\chi$. The other temporal operators and the booleans are treated in the same way.

Now we extend (1) to general $\boldsymbol{CI}$-terms. The only non-trivial case is $\psi = \boldsymbol{I}\chi$, where χ is a $\boldsymbol{CI}$-term. (The case $\psi = \boldsymbol{I}\neg\chi$ is considered analogously.) If $(u,n) \models_{\mathfrak{M}} \boldsymbol{I}\chi$ then $(v,n) \models_{\mathfrak{M}} \chi$ whenever uSv, and so $(u,n) \models_{\mathfrak{K}} \boldsymbol{I}\chi$. Conversely, suppose $(u,n) \models_{\mathfrak{K}} \boldsymbol{I}\chi$, but $(u,n) \not\models_{\mathfrak{M}} \boldsymbol{I}\chi$. Then there is $v \in W$ such that uRv and $(v,n) \not\models_{\mathfrak{M}} \chi$. Take any $w \in V_0$ with vRw. Clearly, $(w,n) \not\models_{\mathfrak{M}} \chi$. But then $(w,n) \not\models_{\mathfrak{K}} \chi$, contrary to uSw and $(u,n) \models_{\mathfrak{K}} \boldsymbol{I}\chi$.

Finally, using this fact we can easily extend (1) to arbitrary $\boldsymbol{CI}$-formulas simply because they are constructed from general $\boldsymbol{CI}$-terms using operators which do not depend on the structure of the underlying **S4**-frame. ❑

By combining the two lemmata above and Theorem 14, we obtain the following:

Theorem 20. (i) *An $\mathcal{ST}_2^+$-formula is satisfiable in a tt-model with FSA iff its modal translation is satisfiable in a Kripke $\mathcal{ML}$-model with FSA whose underlying* **S4***-frame is of depth* ≤ 1.

(ii) *An $\mathcal{ST}_1^+$-formula is satisfiable in a tt-model iff its modal translation is satisfiable in a Kripke $\mathcal{ML}$-model whose underlying* **S4***-frame is of depth* ≤ 1.

In fact, it turns out that even simpler **S4**-frames are enough to satisfy $\mathcal{ST}_2^+$-formulas. We remind the reader that an **S4**-frame $\langle W, R \rangle$ is said to be of *width* n if no point in W has more than n R-incomparable successors.

Renz [27] showed actually that an RCC-8 formula φ is satisfiable iff $\varphi^\dagger$ is satisfiable in a Kripke model based on a frame of depth ≤ 1 and width ≤ 2. We will prove now that this result can be generalized to the temporal case.

Theorem 21. (i) *An $\mathcal{ST}_2^+$-formula φ is satisfiable in a tt-model with FSA iff $\varphi^\dagger$ is satisfiable in a Kripke $\mathcal{ML}$-model with FSA whose underlying* **S4***-frame is of depth* ≤ 1 *and width* ≤ 2.

(ii) *An $\mathcal{ST}_1^+$-formula φ is satisfiable in a tt-model iff $\varphi^\dagger$ is satisfiable in a Kripke $\mathcal{ML}$-model whose underlying* **S4***-frame is of depth* ≤ 1 *and width* ≤ 2.

Proof Suppose φ is satisfied in a tt-model $\mathfrak{M} = \langle \mathfrak{T}, \mathbb{N}, \mathfrak{a} \rangle$ and assume that $X_1, \ldots, X_m$ are all region variables occurring in φ. Denote by Φ the set of all atoms of $\mathcal{ST}_2^+$ that contain only these variables.

For every $n \in \mathbb{N}$, take m fresh variables $X_1^n, \ldots, X_m^n$. Given a region term t built up from $X_1, \ldots, X_m$, define inductively another term t^n in the following way:

- if $t = X_i$, then $t^n = X_i^n$;
- $(t_1 \wedge t_2)^n = t_1^n \wedge t_2^n$;
- $(t_1 \vee t_2)^n = t_1^n \vee t_2^n$;
- $(\neg t)^n = \neg t^n$;
- $(\bigcirc t)^n = t^{n+1}$;
- if $t = \Diamond^+ t_1$, then we take minimal $k_1, \ldots, k_l > n$ such that

  $$\mathfrak{a}(\Diamond^+ t_1, n) = \mathfrak{a}(t_1, k_1) \cup \cdots \cup \mathfrak{a}(t_1.k_l)$$

  (they exist by FSA) and put $t^n = t_1^{k_1} \vee \cdots \vee t_1^{k_l}$;
- for $t = \Box^+ t_1$ we take minimal $k_1, \ldots, k_l > n$ such that

  $$\mathfrak{a}(\Diamond^+ t_1, n) = \boldsymbol{IC}(\mathfrak{a}(t_1, k_1) \cap \cdots \cap \mathfrak{a}(t_1, k_l))$$

  and put $t^n = t_1^{k_1} \wedge \cdots \wedge t_1^{k_l}$.

Now let

$$\Delta_n = \{P(t_1^n, t_2^n) : P(t_1, t_2) \in \Phi \text{ and } (\mathfrak{M}, n) \models P(t_1, t_2)\}$$

and

$$\Delta = \bigcup_{n \in \mathbb{N}} \Delta_n.$$

By definition, Δ contains no occurrences of temporal operators. Denote by Ψ the set of atoms from Φ without temporal operators.

Let $\mathfrak{b}$ be the assignment in $\mathfrak{T}$ defined by $\mathfrak{b}(X_i^n) = \mathfrak{a}(X_i, n)$. It is readily checked by induction that for every region term t we have $\mathfrak{b}(t^n) = \mathfrak{a}(t, n)$, and so

$$\mathfrak{T} \models^{\mathfrak{b}} P(t_1^n, t_2^n) \quad \text{iff} \quad P(t_1^n, t_2^n) \in \Delta_n,$$

for all $P(t_1, t_2) \in \Phi$.

Conversely, suppose we have a topological space $\mathfrak{T}'$ and an assignment $\mathfrak{b}$ in it such that

$$\mathfrak{T}' \models^{\mathfrak{b}} P(t_1^n, t_2^n) \quad \text{iff} \quad P(t_1^n, t_2^n) \in \Delta_n,$$

for all $P(t_1, t_2) \in \Phi$. Construct a tt-model $\mathfrak{M}' = \langle \mathfrak{T}', \mathbb{N}, \mathfrak{a}' \rangle$ by taking, for every $n \in \mathbb{N}$, $\mathfrak{a}'(X_i, n) = \mathfrak{b}(X_i^n)$. Then we shall have $\mathfrak{a}'(t, n) = \mathfrak{b}(t^n)$ and so $(\mathfrak{M}', n) \models P(t_1, t_2)$ iff $P(t_1^n, t_2^n) \in \Delta_n$. It follows that φ is satisfied in $\mathfrak{M}'$.

Thus, to prove our theorem it suffices to construct a Kripke model $\mathfrak{K}' = \langle \mathfrak{F}', \mathfrak{V}' \rangle$ based on a frame $\mathfrak{F}' = \langle W', R' \rangle$ of depth ≤ 1 and width ≤ 2 and such that for every $P(t_1, t_2)$,

$$\mathfrak{K}' \models (P(t_1, t_2))^{\dagger} \quad \text{iff} \quad P(t_1, t_2) \in \Delta.$$

By Theorem 20, we have a model $\mathfrak{K} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ such that $\mathfrak{F} = \langle W, R \rangle$ is of depth ≤ 1 and for every $P(t_1, t_2)$,

$$\mathfrak{K} \models (P(t_1, t_2))^\dagger \quad \text{iff} \quad P(t_1, t_2) \in \Delta.$$

Denote by $\Sigma_\exists$ the set of formulas of the form $\exists \chi$ and their negations which are true in $\mathfrak{K}$ and such that $\exists \chi$ is a subformula of $(P(t_1, t_2))^\dagger$, for some $P(t_1, t_2) \in \Psi$.

For every $\exists \chi \in \Sigma_\exists$ there is $x_\chi \in W$ such that $x_\chi \models \chi$. We may assume that we have chosen mutually different x_χ and that all x of depth 1 (so, those with proper successors) are of the form x_χ for some $\exists \chi \in \Sigma_\exists$. Moreover, we may assume that no point has more than 1 proper predecessor.

We construct $\mathfrak{F}'$ in the following way.

First, for every $\exists \chi \in \Sigma_\exists$ such that there exists a point of depth 0 with $x \models \chi$, we remove the point x_χ from W whenever it has depth 1. Let W' be the resulting set of worlds.

Suppose now that $\exists \chi \in \Sigma_\exists$ and $x_\chi \in W'$. By definition, χ has one of the following forms:

$$t_1^* \wedge t_2^*, \quad t_1^* \wedge \boldsymbol{C} \neg t_2^*, \quad \boldsymbol{I} t_1^* \wedge \boldsymbol{I} t_2^*, \quad \neg t_1^* \wedge t_2^*, \quad \neg t_1^* \wedge \boldsymbol{I} t_2^*.$$

And without loss of generality we may assume that each t_i^* is of the form $\boldsymbol{CI}\psi_i$.

Depending on the form of χ we delete some of the R-arrows coming from x_χ. There are four possible cases.

Case 1: $\chi = \boldsymbol{CI}\psi_1 \wedge \boldsymbol{CI}\psi_2$. Then we select two points $x_1, x_2 \in W'$ of depth 0 such that $x_i \models \boldsymbol{CI}\psi_i$ and $x_\chi R x_i$, for $i = 1, 2$, and remove all R-arrows leading from x_χ to points different from x_1, x_2.

Case 2: $\chi = \boldsymbol{CI}\psi_1 \wedge \boldsymbol{C}\neg\boldsymbol{CI}\psi_2$. Then we select $x_1, x_2 \in W'$ of depth 0 such that $x_1 \models \boldsymbol{CI}\psi_1$, $x_2 \models \neg\boldsymbol{CI}\psi_2$, $x_\chi R x_i$, and remove all arrows leading from x_χ to points different from x_1, x_2.

Case 3: $\chi = \boldsymbol{CI}\psi_1 \wedge \neg\boldsymbol{CI}\psi_2$. Then we select $x_1, x_2 \in W'$ of depth 0 such that $x_1 \models \boldsymbol{CI}\psi_1$, $x_2 \models \neg\boldsymbol{CI}\psi_2$, $x_\chi R x_i$, and remove all arrows leading from x_χ to points different from x_1, x_2.

Case 4: Otherwise take arbitrary $x_1, x_2 \in W'$ of depth 0 (if they exist) such that $x_\chi R x_i$ and remove all arrows leading from x_χ to points different from x_1, x_2.

Denote by R' the resulting relation and define a valuation $\mathfrak{V}'$ in $\mathfrak{F}' = \langle W', R' \rangle$ by taking, for every variable p, $x \in \mathfrak{V}'(p)$ iff there is $y \in W'$ of depth 0 such that $xR'y$ and $y \in \mathfrak{V}(p)$.

Let us show that the model $\mathfrak{M}' = \langle \mathfrak{F}', \mathfrak{V}' \rangle$ is as required.

Clearly, the points of depth 0 in $\mathfrak{M}$ and $\mathfrak{M}'$ validate the same formulas (without $\exists$). By the construction, we also have that if $\mathfrak{M} \models \exists \chi$ then $\mathfrak{M}' \models \exists \chi$. So it remains to show that if $\mathfrak{M} \models \neg \exists \chi$ then $\mathfrak{M}' \models \neg \exists \chi$. Consider all five possible cases for χ.

(1) $\chi = \boldsymbol{CI}\psi_1 \wedge \boldsymbol{CI}\psi_2$. If $(\mathfrak{M}', x) \models \chi$ then there are $x_1, x_2 \in W'$ of depth 0 such that $xR'x_i$ and $x_i \models \boldsymbol{CI}\psi_i$. But then $(\mathfrak{M}, x) \models \chi$.

(2) $\chi = \boldsymbol{CI}\psi_1 \wedge \boldsymbol{C}\neg\boldsymbol{CI}\psi_2$. Let $(\mathfrak{M}', x) \models \chi$. Then $y \models \boldsymbol{CI}\psi_1$ for some y of depth 0, $xR'y$. Thus $(\mathfrak{M}, x) \models \boldsymbol{CI}\psi_1$, and so $(\mathfrak{M}, x) \models \neg\boldsymbol{C}\neg\boldsymbol{CI}\psi_2$ which means that $z \models \boldsymbol{CI}\psi_2$ whenever xRz. So we have $(\mathfrak{M}', z) \models \boldsymbol{CI}\psi_2$ whenever $xR'z$, contrary to $(\mathfrak{M}', x) \models \boldsymbol{C}\neg\boldsymbol{CI}\psi_2$.

(3) $\chi = \boldsymbol{ICI}\psi_1 \wedge \boldsymbol{ICI}\psi_2$. Suppose $(\mathfrak{M}', x) \models \chi$. Then for every successor y of x of depth 0, we have $(\mathfrak{M}', y) \models \chi$ and so $(\mathfrak{M}, y) \models \chi$, which is a contradiction.

(4) $\chi = \neg \boldsymbol{CI}\psi_1 \wedge \boldsymbol{CI}\psi_2$. If $(\mathfrak{M}', x) \models \chi$ then $(\mathfrak{M}', y) \models \boldsymbol{CI}\psi_2$ for some y of depth 0, $xR'y$. But then $(\mathfrak{M}', y) \models \neg\psi_1$, so $(\mathfrak{M}, y) \models \chi$, which is again a contradiction.

(5) $\chi = \neg \boldsymbol{CI}\psi_1 \wedge \boldsymbol{ICI}\psi_2$. This case is considered in the same way. ❑

5 Decidability

We begin by considering the logics $\mathcal{ST}_0$ and $\mathcal{ST}_1$. To simplify presentation, we assume that we have only one temporal operator $\mathcal{U}$. the reader should have no problems in extending the proofs to the language with $\mathcal{S}$.

Suppose $\varphi \in \mathcal{ST}_1$. Denote by $term(\varphi)$ the set of region terms $\bigcirc^m X$ occurring in φ. And let $sub(\varphi)$ be the set of all subformulas in φ. Without loss of generality we may assume that $sub(\varphi)$ is closed under $\neg$.

Definition 22 (fork). By a *fork* we mean a frame $\mathfrak{f} = \langle W_\mathfrak{f}, R_\mathfrak{f} \rangle$ such that

- $W_\mathfrak{f} = \{a_\mathfrak{f}, b_\mathfrak{f}, c_\mathfrak{f}\}$,
- $a_\mathfrak{f}$ is the root of $\mathfrak{f}$, while $b_\mathfrak{f}$ and $c_\mathfrak{f}$ are its two immediate successors.

Thus $R_\mathfrak{f}$ is the reflexive closure of the relation

$$\{\langle a_\mathfrak{f}, b_\mathfrak{f} \rangle, \langle a_\mathfrak{f}, c_\mathfrak{f} \rangle\}.$$

A *labelled fork* for φ is the pair $\langle \mathfrak{f}, l_\mathfrak{f} \rangle$, where $\mathfrak{f}$ is a fork and $l_\mathfrak{f}$ a *labelling function* which associates with every point x in $\mathfrak{f}$ a subset $l_\mathfrak{f}(x)$ of $term(\varphi)$ in such a way that $t \in l_\mathfrak{f}(x)$ iff there is a final point y in $\mathfrak{f}$ accessible from x with $t \in l_\mathfrak{f}(y)$.

A set $\boldsymbol{F}$ of labelled forks can be regarded as a Kripke frame $\langle \boldsymbol{W}, \boldsymbol{R} \rangle$—the disjoint union of forks in $\boldsymbol{F}$—with all points $x \in \boldsymbol{W}$ labelled by $l_{\boldsymbol{F}}(x) \subseteq term(\varphi)$, where $l_{\boldsymbol{F}}(x) = l_\mathfrak{f}(x)$ if $x \in W_\mathfrak{f}$.

For an RCC-8 predicate P and region terms t_1, t_2, we write $\boldsymbol{F} \models P(t_1, t_2)$ iff $P(t_1, t_2)$ holds in the topological space determined by $\boldsymbol{F}$ under the assignment

$$\mathfrak{a}(t) = \{x \in \boldsymbol{W} : t \in l_{\boldsymbol{F}}(x)\}.$$

Definition 23 (type). A *formula type* for φ is a subset Φ of $sub(\varphi)$ such that

- $\psi \wedge \chi \in \Phi$ iff $\psi, \chi \in \Phi$, for every $\psi \wedge \chi \in sub(\varphi)$;
- $\neg\psi \in \Phi$ iff $\psi \notin \Phi$, for every $\psi \in sub(\varphi)$.

Definition 24 (quasistate). We say that the pair $\langle \boldsymbol{F}, \Phi \rangle$ is a *quasistate* for φ if

- $\boldsymbol{F}$ is a set of pairwise non-isomorphic labelled forks, and
- Φ is a formula type for φ such that for every $P(t_1, t_2) \in sub(\varphi)$, we have $P(t_1, t_2) \in \Phi$ iff $\boldsymbol{F} \models P(t_1, t_2)$.

Denote by $\sharp(\varphi)$ the number of pairwise non-isomorphic quasistates for φ. Clearly

$$\sharp(\varphi) \leq 2^{2^{2 \cdot |term(\varphi)|}} \cdot 2^{|sub(\varphi)|}.$$

Definition 25 (suitable pair). Say that an ordered pair $\langle \mathfrak{f}, l_\mathfrak{f}\rangle$, $\langle \mathfrak{g}, l_\mathfrak{g}\rangle$ of labelled forks for φ is *suitable* if there is an isomorphism σ from $\mathfrak{f}$ onto $\mathfrak{g}$ such that, for all $\bigcirc t \in term(\varphi)$ and all $x \in W_\mathfrak{f}$,

$$\bigcirc t \in l_\mathfrak{f}(x) \quad \text{iff} \quad t \in l_\mathfrak{g}(\sigma(x)).$$

An ordered pair $\langle \boldsymbol{F}, \Phi\rangle$, $\langle \boldsymbol{G}, \Psi\rangle$ of quasistates for φ is called *suitable* if

- for every $\langle \mathfrak{f}, l_\mathfrak{f}\rangle \in \boldsymbol{F}$ there is $\langle \mathfrak{g}, l_\mathfrak{g}\rangle \in \boldsymbol{G}$ such that the pair $\langle \mathfrak{f}, l_\mathfrak{f}\rangle$, $\langle \mathfrak{g}, l_\mathfrak{g}\rangle$ is suitable;
- for every $\langle \mathfrak{g}, l_\mathfrak{g}\rangle \in \boldsymbol{G}$ there is $\langle \mathfrak{f}, l_\mathfrak{f}\rangle \in \boldsymbol{F}$ such that $\langle \mathfrak{f}, l_\mathfrak{f}\rangle$, $\langle \mathfrak{g}, l_\mathfrak{g}\rangle$ is a suitable pair.

Definition 26 (state function). A *state function* for φ is a function I which associates with each $n \in \mathbb{N}$ a quasistate $I(n) = \langle \boldsymbol{F}_n, \Phi_n\rangle$ for φ.

Definition 27 (quasimodel). A state function I with $I(n) = \langle \boldsymbol{F}_n, \Phi_n\rangle$ is called a *quasimodel* for φ if

- the pair $I(m)$, $I(m+1)$ of quasistates is suitable for every $m \in \mathbb{N}$,
- $\chi \mathcal{U} \psi \in \Phi_n$ iff there exists $k > n$ such that $\psi \in \Phi_k$ and $\chi \in \Phi_l$ for all $l \in (n, k)$ and $n \in \mathbb{N}$.

Say that φ is *satisfied* in the quasimodel I if $\varphi \in \Phi_n$ for some $n \in \mathbb{N}$.

Theorem 28. *A formula $\varphi \in \mathcal{ST}_1$ is satisfied in a tt-model iff it is satisfied in a quasimodel for φ.*

Proof ($\Rightarrow$) By Theorem 21, we may assume that φ is satisfied in a tt-model $\mathfrak{M} = \langle \mathfrak{T}, \mathbb{N}, \mathfrak{a}\rangle$ the underlying topological space $\mathfrak{T}$ of which is determined by a disjoint union $\mathfrak{F} = \langle W, R\rangle$ of forks.

For every $n \in \mathbb{N}$, define an equivalence relation $\sim_n$ on the set of forks in $\mathfrak{F}$ by taking $\mathfrak{f} \sim_n \mathfrak{g}$ iff there is an isomorphism σ from $\mathfrak{f}$ onto $\mathfrak{g}$ such that $x \in \mathfrak{a}(t, n)$ iff $\sigma(x) \in \mathfrak{a}(t, n)$, for all $t \in term(\varphi)$ and all x in $\mathfrak{f}$.

Pick a representative of every $\sim_n$-equivalence class and define $\boldsymbol{F}_n$ to be the set of all selected representatives labelled in accordance with $\mathfrak{a}$. Let Φ_n be the set of all formulas in $sub(\varphi)$ that hold in $\mathfrak{M}$ at moment n.

It is easy to check that the state function $I(n) = \langle \boldsymbol{F}_n, \Phi_n\rangle$ is a quasimodel satisfying φ.

($\Leftarrow$) Conversely, suppose φ is satisfied in a quasimodel I for φ. Say that a function r which associates with every $n \in \mathbb{N}$ a labelled fork $r(n)$ in $I(n)$ is a *run* through I if for every $m \in \mathbb{N}$, the pair of $r(m)$, $r(m+1)$ is suitable. Let $\mathcal{R}$ be the set of all runs through I.

Define a topological temporal model $\mathfrak{M} = \langle \mathfrak{T}, \mathbb{N}, \mathfrak{a}\rangle$ as follows. The topological space $\mathfrak{T}$ is determined by the frame $\mathfrak{F} = \langle W, R\rangle$, where

$$W = \mathcal{R} \times \{a, b, c\}$$

and, for all $x, y \in W$

$$xRy \quad \text{iff} \quad x = y \,\vee\, \exists r \in \mathcal{R}\ (x = \langle r, a\rangle \ \&\ (y = \langle r, b\rangle \,\vee\, y = \langle r, c\rangle)).$$

And the assignment $\mathfrak{a}$ is defined by

$$\mathfrak{a}(X,n) = \{\langle r,d\rangle : X \in l_{r(n)}(d_{r(n)})\}.$$

It is not hard to check by induction that φ is satisfied in $\mathfrak{M}$. ❑

We are going to prove now that, given an $\mathcal{ST}_1$-formula φ, we can effectively recognize whether there exists a quasimodel satisfying φ. The idea of the proof is to show that a satisfiable φ can always be satisfied in a 'periodical' quasimodel of the form

$$I(0),\ldots,I(k),(I(k+1),\ldots,I(l))^*$$

with effectively bounded k and l.

Let us first fix some notation concerning sequences (of arbitrary elements). Given a sequence $s = s(0), s(1), \ldots$ and $i \geq 0$, we denote by $s^{\leq i}$ and $s^{>i}$ the head $s(0),\ldots,s(i)$ and the tail $s(i+1), s(i+2),\ldots$ of s, respectively; $s_1 * s_2$ is the concatenation of sequences s_1 and s_2; $|s|$ denotes the length of s and

$$s^* = s * s * s * \ldots$$

Lemma 29. *Let $I = I(0), I(1), \ldots$ be a quasimodel for φ and $I(n) = I(m)$ for some $n < m$. Then $I_{nm} = I^{\leq n} * I^{>m}$ is also a quasimodel for φ.*

If a subsequence of a quasimodel I for φ is a quasimodel for φ itself, then we call it a *subquasimodel* of I. For example, I_{nm} in Lemma 29 is a subquasimodel of I.

Lemma 30. *Every quasimodel I for φ contains a subquasimodel $I_1 * I_2$ such that $|I_1| \leq \sharp(\varphi)$ and each quasistate in I_2 occurs in this sequence infinitely many times.*

Let, as before, $I(n) = \langle \boldsymbol{F}_n, \Phi_n\rangle$. Say that a formula $\psi\mathcal{U}\chi \in \Phi_n$ is *realized in m steps* in I if there is $l \in (0, m+1)$ such that $\chi \in \Phi_{n+l}$ and $\psi \in \Phi_{n+k}$ for all $k \in (0,l)$.

Lemma 31. *Let $I = I_1 * I_2$ be a quasimodel for φ (with quasistates of the form $\langle \boldsymbol{F}_i, \Phi_i\rangle$, $i \in \mathbb{N}$) satisfying the requirements of Lemma 30 and let $n = |I_1| + 1$. Then I contains a subquasimodel of the form $I_1 * I_0 * I_2^{>l}$, for some $l \geq 0$, such that*

(i) *$|I_0| \leq |sub(\varphi)| \cdot \sharp(\varphi) + \sharp(\varphi)$;*

(ii) *every formula $\psi\mathcal{U}\chi \in \Phi_n$ is realized in $|I_0|$ steps;*

(iii) *$I_0(0) = I_2^{>l}(0)$.*

Proof Suppose $\psi\mathcal{U}\chi \in \Phi_n$, Then there exists $m > 0$ such that $\chi \in \Phi_{n+m}$ and $\psi \in \Phi_{n+k}$ for all $k \in (0,m)$. Assume now that $0 < i < j < m$ and $I(n+i) = I(n+j)$. In view of Lemma 29, $I_1 * I_2^{\leq i} * I_2^{>j}$ is a subquasimodel of I. It follows that we can construct a subquasimodel $I_1 * I_2^{\leq 1} * I_3$ of I in which $\psi\mathcal{U}\chi$ is realized in $m_1 \leq \sharp(\varphi)$ steps.

Then we consider another formula $\psi'\mathcal{U}\chi' \in \Phi_n$ and assume that it is realized in $m_2 > m_1$ steps. Using Lemma 29 once again (and deleting repeating quasistates in the interval $I_3(m_1),\ldots,I_3(m_2)$) we select a subquasimodel $I_1 * I_2^{\leq 1} * I_3^{\leq m_1} * I_4$ of I which realizes both $\psi\mathcal{U}\chi$ and $\psi'\mathcal{U}\chi'$ in $2\sharp(\varphi)$ steps.

Having analyzed all distinct formulas of the form $\psi\mathcal{U}\chi \in \Phi_n$ we obtain a subquasimodel $I_1 * I_2^{\leq 1} * I'$ of I which realizes all those formulas in $|sub(\varphi)| \cdot \sharp(\varphi)$ steps. And $\leq \sharp(\varphi)$ additional quasistates may be required to comply with (iii). ❏

Lemma 32. *Suppose I_1 and I_2 are finite sequences of quasistates for φ of length l_1 and l_2, respectively, and let*

$$I = I_1 * I_2^*$$

with $I(n) = \langle \boldsymbol{F}_n, \Phi_n \rangle$. Then I is a quasimodel for φ whenever the following conditions hold:

1. *for every $i \leq l_1 + l_2$, the pair $\boldsymbol{F}_i$, $\boldsymbol{F}_{i+1}$ is suitable;*
2. *for every $i \leq l_1 + l_2$, and every formula $\psi\mathcal{U}\chi \in sub(\varphi)$,*

$$\psi\mathcal{U}\chi \in \Phi_i \text{ iff either } \chi \in \Phi_{i+1} \text{ or } \psi \in \Phi_{i+1} \text{ and } \psi\mathcal{U}\chi \in \Phi_{i+1};$$

3. *for every $i \leq l_1 + 1$, all formulas of the form $\psi\mathcal{U}\chi \in \Phi_i$ are realized in $l_1 + l_2 - i$ steps.*

As a consequence of the two preceding lemmas we immediately obtain

Theorem 33. *An $\mathcal{ST}_1$-formula φ is satisfiable in a topological temporal model iff there are two sequences I_1 and I_2 of quasistates for φ such that $I = I_1 * I_2^*$ satisfies conditions 1–3 of Lemma 32, all quasistates in I_1 are distinct (and so $|I_1| \leq \sharp(\varphi)$),*

$$|I_2| \leq |sub(\varphi)| \cdot \sharp(\varphi) + \sharp(\varphi),$$

and $\varphi \in I(1)$.

This theorem provides us with an EXPSPACE algorithm which is capable of deciding whether a given $\mathcal{ST}_1$-formula φ is satisfiable in a quasimodel for φ. Here is a rough description of such an algorithm; in view of the equality EXPSPACE = NEXPSPACE, it can be non-deterministic.

First, we guess $l_1 \leq \sharp(\varphi)$ and $l_2 \leq |sub(\varphi)| \cdot \sharp(\varphi) + \sharp(\varphi)$ and write them in binary using thereby exponential space in $\ell(\varphi)$. Then we guess a set $\boldsymbol{F}_0$ of $\leq 2^{2 \cdot |term(\varphi)|}$ labelled forks, a subset $\Phi_0 \subseteq sub(\varphi)$ containing φ, and check that $\langle \boldsymbol{F}_0, \Phi_0 \rangle$ is a quasistate. In the same way we guess a quasistate $\langle \boldsymbol{F}_1, \Phi_1 \rangle$ (here we do not need the condition $\varphi \in \Phi_1$) and check whether the pair $\langle \boldsymbol{F}_0, \Phi_0 \rangle$, $\langle \boldsymbol{F}_1, \Phi_1 \rangle$ is suitable and whether condition 2 of Lemma 32 is satisfied. After that we remove $\langle \boldsymbol{F}_0, \Phi_0 \rangle$, guess $\langle \boldsymbol{F}_2, \Phi_2 \rangle$, check the pair $\langle \boldsymbol{F}_1, \Phi_1 \rangle$, $\langle \boldsymbol{F}_2, \Phi_2 \rangle$, and so on till we reach $\langle \boldsymbol{F}_{l_1+1}, \Phi_{l_1+1} \rangle$—this quasistate is stored in memory together with the set Σ of all formulas of the form $\chi\mathcal{U}\psi$. We proceed further in the same way as before deleting $\chi\mathcal{U}\psi$ from Σ every time we reach Φ_i containing ψ. If the pair $\langle \boldsymbol{F}_{l_1+l_2}, \Phi_{l_1+l_2} \rangle$, $\langle \boldsymbol{F}_{l_1+1}, \Phi_{l_1+1} \rangle$ is suitable and Σ is empty, then φ is satisfiable. This proves Theorem 4 (i).

Suppose now that φ is an $\mathcal{ST}_0$-formula. It follows from the proof of Theorem 21 that φ is satisfiable iff it is satisfied in a quasimodel all quasistates in which contain $\leq c \cdot \ell(\varphi)$ labelled forks, $c = const$. Thus

$$\sharp(\varphi) \leq c \cdot \ell(\varphi) \cdot 2^{2 \cdot |term(\varphi)|} \cdot 2^{|sub(\varphi)|},$$

and so the satisfiability problem for $\mathcal{ST}_0$-formulas is decidable in PSPACE. Recall that the satisfiability problem for pure **PTL**-formulas is PSPACE-complete (see e.g. [14]). Given such a formula ϕ, we replace every propositional variable p in it with the RCC-8 predicate $\mathsf{EQ}(X^p, Y^p)$ thus obtaining an $\mathcal{ST}_0$-formula ϕ°. It should be clear that ϕ is satisfiable (in a model for **PTL**) iff ϕ° is satisfiable (in a tt-model). Consequently, the satisfiability problem for $\mathcal{ST}_0$-formulas is PSPACE-complete. This proves Theorem 3.

As to $\mathcal{ST}_1'$-formulas, it is not hard to see that such a formula φ is satisfiable in a tt-model iff it is satisfied in a model $\langle \mathfrak{T}, \mathfrak{L}, \mathfrak{a} \rangle$, where $\mathfrak{L}$ is a strict linear order with $\leq \ell(\varphi)$ points and $\mathfrak{T}$ is determined by a set of $\leq (\ell(\varphi))^2$ forks. This yields a satisfiability checking algorithm which is in NP, and thus proves Theorem 4 (ii).

Let us consider now $\mathcal{ST}_2$-formulas and assume again that we have only one temporal operator $\mathcal{U}$. Suppose $\varphi \in \mathcal{ST}_2$. As before $term(\varphi)$ is the set of region terms occurring in φ; now besides region variables it may contain. terms of the form $\Diamond^+ X$, $\Box^+ X$, and $\bigcirc X$.

Definition 34 (run). Suppose I is a state function for φ. By a *run* r in I we mean a function $r = \langle r_1, r_2, r_3 \rangle$ with domain $\mathbb{N}$ such that, for all $n \in \mathbb{N}$, there exists a fork $\mathfrak{f} = \langle W_\mathfrak{f}, R_\mathfrak{f} \rangle$ underlying some labelled fork in $\boldsymbol{F}_n$ such that $r_1(n)$ is the root of $\mathfrak{f}$ and $r_2(n), r_3(n)$ are its immediate successors, and for all $i \in \{1, 2, 3\}$ and $n \in \mathbb{N}$,

- $\bigcirc t \in l_\mathfrak{f}(r_i(n))$ iff $t \in l_\mathfrak{g}(r_i(n+1))$,
- $\Box^+ t \in l_\mathfrak{f}(r_i(n))$ iff, for all $m > n$, $t \in l_\mathfrak{g}(r_i(m))$,
- $\Diamond^+ t \in l_\mathfrak{f}(r_i(n))$ iff there exists $m > n$ such that $t \in l_\mathfrak{g}(r_i(m))$.

Definition 35 (FSA-quasimodel). A pair $\langle I, \mathcal{R} \rangle$ consisting of a state function I with $I(n) = \langle \boldsymbol{F}_n, \Phi_n \rangle$ and a *finite* set of runs $\mathcal{R}$ in I is called a *FSA–quasimodel* for φ if

- $\chi \mathcal{U} \psi \in I(n)$ iff there exists $k > n$ such that $\psi \in \Phi_k$ and $\chi \in \Phi_l$ for all $l \in (n, k)$ and $n \in \mathbb{N}$.
- for all $n \in \mathbb{N}$ and fork $\mathfrak{f}$ underlying a labelled fork in $\boldsymbol{F}_n$ there exists $r \in \mathcal{R}$ such that $W_\mathfrak{f} = \{r_1(n), r_2(n), r_3(n)\}$.

Say that φ is *satisfied* in the quasimodel I if $\varphi \in \Phi_n$ for some $n \in \mathbb{N}$.

Theorem 36. *The following conditions are equivalent:*

1. *A formula $\varphi \in \mathcal{ST}_2$ is satisfied in a tt-model with FSA,*
2. *φ is satisfied in a FSA–quasimodel for φ.*
3. *φ is satisfied in a tt-model with finite domain.*

Proof (1) $\Rightarrow$ (2). By Theorem 21, we may assume that φ is satisfied in a tt-model $\mathfrak{M} = \langle \mathfrak{T}, \mathbb{N}, \mathfrak{a} \rangle$ with FSA the underlying topological space $\mathfrak{T}$ of which is determined by a disjoint union $\mathfrak{F} = \langle W, R \rangle$ of forks.

For every $n \in \mathbb{N}$, define an equivalence relation $\sim_n$ on the set of forks in $\mathfrak{F}$ by taking $\mathfrak{f} \sim_n \mathfrak{g}$ iff there is an isomorphism σ from $\mathfrak{f}$ onto $\mathfrak{g}$ such that $x \in \mathfrak{a}(t, n)$ iff

$\sigma(x) \in \mathfrak{a}(t,n)$, for all $t \in term(\varphi)$ and all x in $\mathfrak{f}$. Observe that, because of FSA, the set $\{\sim_n : n \in \mathbb{N}\}$ is finite. Pick a representative of every $\sim_n$-equivalence class (such that the representatives coincide whenever $\sim_n = \sim_m$) and define $\boldsymbol{F}_n$ to be the set of all selected representatives labelled in accordance with $\mathfrak{a}$.

Let $\mathfrak{f}$ be a fork in $\mathfrak{F}$. Define $r^{\mathfrak{f}} = \left\langle r_1^{\mathfrak{f}}, r_2^{\mathfrak{f}}, r_3^{\mathfrak{f}} \right\rangle$ as follows: for $n \in \mathbb{N}$, let σ_n be the isomorphism from $\mathfrak{f}$ onto the representative $\mathfrak{f}^n$ of $\mathfrak{f}$ in $\boldsymbol{F}_n$ such that $x \in \mathfrak{a}(t,n)$ iff $\sigma_n(x) \in \mathfrak{a}(t,n)$, for all t and $x \in W_{\mathfrak{f}}$. Put

- $r_1^{\mathfrak{f}}(n) = \sigma_n(a_{\mathfrak{f}})$,
- $r_2^{\mathfrak{f}}(n) = \sigma_n(b_{\mathfrak{f}})$,
- $r_3^{\mathfrak{f}}(n) = \sigma_n(c_{\mathfrak{f}})$

It is not difficult to see that all $r^{\mathfrak{f}}$ are runs and that

$$\mathcal{R} = \{r^{\mathfrak{f}} : \mathfrak{f} \text{ a fork in } \mathfrak{F}\}$$

is finite. Let Φ_n be the set of all formulas in $sub(\varphi)$ that hold in $\mathfrak{M}$ at moment n. It is easy to check that the pair $\langle I, \mathcal{R}\rangle$ with $I(n) = \langle \boldsymbol{F}_n, \Phi_n\rangle$ is a FSA-quasimodel satisfying φ.

(2) $\Leftarrow$ (3). Suppose φ is satisfied in a FSA-quasimodel $\langle I, \mathcal{R}\rangle$ for φ.

Define a topological temporal model $\mathfrak{M} = \langle \mathfrak{T}, \mathbb{N}, \mathfrak{a}\rangle$ as follows. The topological space $\mathfrak{T}$ is determined by the frame $\mathfrak{F} = \langle W, R\rangle$, where

$$W = \mathcal{R} \times \{a, b, c\}$$

and, for all $x, y \in W$

$$xRy \quad \text{iff} \quad x = y \;\vee\; \exists r \in \mathcal{R}\; (x = \langle r, a\rangle \;\&\; (y = \langle r, b\rangle \;\vee\; y = \langle r, c\rangle))\,.$$

And the assignment $\mathfrak{a}$ is defined by

$$\mathfrak{a}(X, n) = \{\langle r, d\rangle : X \in l_{r(n)}(d_{r(n)})\}.$$

It is not hard to check by induction that φ is satisfied in $\mathfrak{M}$. Clearly W is finite.

(3) $\Rightarrow$ (1) is trivial. ❑

We fix a formula $\varphi \in \mathcal{ST}_2$ and an enumeration $\left\langle \mathfrak{h}_1, \ldots, \mathfrak{h}_{n_\varphi} \right\rangle$ of all labelled forks for φ, $n_\varphi \leq 2^{2 \cdot |term(\varphi)|}$.

Let $\langle I, \mathcal{R}\rangle$ be a FSA-quasimodel for φ with $I(n) = \langle \boldsymbol{F}_n, \Phi_n\rangle$. In what follows we write $r(i) = r(j)$ for a run $r = \langle r_1, r_2, r_3\rangle \in \mathcal{R}$ and $i, j \in \mathbb{N}$ whenever

- $\langle r_1(i), r_2(i), r_3(i)\rangle = \langle r_1(j), r_2(j), r_3(j)\rangle$ and
- $l_{\boldsymbol{F}_i}(r_k(i)) = l_{\boldsymbol{F}_j}(r_k(j))$, for all $k \in \{1, 2, 3\}$.

We write $r(i) = \mathfrak{h}$ whenever $\mathfrak{h}$ is the labelled fork in $\boldsymbol{F}_i$ based on $\{r_1(i), r_2(i), r_3(i)\}$.

Define an equivalence relation $\sim_{\mathcal{R}}$ on $\mathbb{N}$ by taking $i \sim_{\mathcal{R}} j$ iff

- $I(i) = I(j)$ and
- $\forall r \in \mathcal{R}\; r(i) = r(j)$.

Denote by $[n]_{\mathcal{R}}$ the $\sim_{\mathcal{R}}$-equivalence class generated by n.

Besides, for each $n \in \mathbb{N}$, we define one more equivalence relation $\sim^n_{\mathcal{R}}$ on $\mathbb{N}$ by taking $i \sim^n_{\mathcal{R}} j$ iff $I(i) = I(j)$ and

- for every $r \in \mathcal{R}$ there is $r' \in \mathcal{R}$ such that $r(n) = r'(n)$ and $r(i) = r'(j)$,
- for every $r \in \mathcal{R}$ there is $r' \in \mathcal{R}$ such that $r(n) = r'(n)$ and $r(j) = r'(i)$.

Lemma 37. *For every $n \in \mathbb{N}$, the number of pairwise distinct $\sim^n_{\mathcal{R}}$-equivalence classes does not exceed*

$$\natural(\varphi) \leq \sharp(\varphi) \cdot 2^{2 \cdot 2^{2 \cdot |term(\varphi)|}}.$$

Proof Fix some $n \in \mathbb{N}$ and define a function $\sigma_i(k,l)$, for $i \in \mathbb{N}$, $k,l \leq n_\varphi$, by taking

$$\sigma_i(k,l) = \begin{cases} 1 & \text{if } \exists r \in \mathcal{R}\ r(n) = \mathfrak{h}_k \ \&\ r(i) = \mathfrak{h}_l, \\ 0 & \text{otherwise.} \end{cases}$$

We then have $i \sim^n_{\mathcal{R}} j$ whenever $I(i) = I(j)$ and $\sigma_i(k,l) = \sigma_j(k,l)$, for all $k,l \leq n_\varphi$. It remains to observe that the number of functions from $\{1,\dots,n_\varphi\}^2$ into $\{0,1\}$ is $2^{n^2_\varphi}$. ❑

The following is easily proved:

Lemma 38. *Every FSA-quasimodel $\langle I, \mathcal{R}\rangle$ for φ contains a FSA-quasimodel $\langle I_1 * I_2, \mathcal{Q}\rangle$ such that $|I_1| \leq \sharp(\varphi)$ and $[n]_{\mathcal{Q}}$ is infinite, for every $n \geq |I_1|$.*

Lemma 39. *Let $\langle I, \mathcal{R}\rangle$ be a quasimodel for φ, $n < i < j$, and $i \sim^n_{\mathcal{R}} j$. Then $\langle I^{\leq i} * I^{>j}, \mathcal{Q} = \mathcal{R}^{\leq i} *_n \mathcal{R}^{>j}\rangle$ is also a FSA-quasimodel for φ, where*

$$\mathcal{R}^{\leq i} *_n \mathcal{R}^{>j} = \{r_1^{\leq i} * r_2^{>j} : r_1, r_2 \in \mathcal{R}, r_1(i) = r_2(j), r_1(n) = r_2(n)\}.$$

Moreover, for all $n' > j$, if $n \sim_{\mathcal{R}} n'$ then $n \sim_{\mathcal{Q}} n' - (j-i)$.

Proof Follows immediately from the definition of $i \sim^n_{\mathcal{R}} j$. ❑

Lemma 40. *Let $\langle I = I_1 * I_2, \mathcal{R}\rangle$ be a FSA-quasimodel for φ (in which we have $I(n) = \langle F_n, \Phi_n\rangle$) such that $n = |I_1| \leq \sharp(\varphi)$ and $[m]_{\mathcal{R}}$ is infinite for all $m > n$. Then $\langle I, \mathcal{R}\rangle$ contains a FSA-subquasimodel of the form $\langle I_1 * I_0 * I_2^{>l}, \mathcal{Q}\rangle$, for some $l \geq 0$, such that*

(i) $|I_0| \leq \natural(\varphi) \cdot (2^{4 \cdot |term(\varphi)|} \cdot 3 \cdot |term(\varphi)| + 1 + 2^{2 \cdot |sub(\varphi)|} \cdot |sub(\varphi)|)$;

(ii) *for every $\mathfrak{f} = \langle W_{\mathfrak{f}}, l_{\mathfrak{f}}\rangle \in \boldsymbol{F}_n$ there is a run $r \in \mathcal{Q}$ through $\mathfrak{f}$ realizing all terms of the form $\Diamond^+ X$ in $\bigcup_{x \in W_{\mathfrak{f}}} l_{\mathfrak{f}}(x)$ in $|I_0|$ steps;*

(iii) *every $\psi_1 \mathcal{U} \psi_2 \in \Phi_n$ is realized in $|f_0|$ steps;*

(iv) $n \sim_{\mathcal{Q}} |I_1 * I_0|$;[2]

(v) *for every labelled fork $\mathfrak{f} \in \boldsymbol{F}_n$ there is a run $r \in \mathcal{Q}$ through $\mathfrak{f}$.*

[2] Note that $I(n) = I_0(0) = I_2^{>l}(0) = I(|I_1 * I_0|)$.

Proof Suppose $\mathfrak{f} = \langle W_\mathfrak{f}, l_\mathfrak{f}\rangle \in \boldsymbol{F}_n$, $\Diamond^+ X \in l_\mathfrak{f}(x)$, for some $x \in W_\mathfrak{f}$, and r is a run through $\mathfrak{f}$. Take $k \in \{1,2,3\}$ such that $x = r_k(n)$.

Then there exists $m > 0$ such that $X \in l_{\boldsymbol{F}_{n+m}}(r_k(n+m))$. Assume now that $0 < i < j < m$, $r(n+i) = r(n+j)$ and $n+i \sim_n^{\mathcal{R}} n+j$. In view of Lemma 39,

$$\left\langle I_1 * I_2^{\leq i} * I_2^{>j}, \mathcal{Q}_0 = \mathcal{R}^{\leq n+i} *_n \mathcal{R}^{>n+j} \right\rangle$$

is a FSA-subquasimodel of $\langle I, \mathcal{R}\rangle$, $r^{\leq n+i} * r^{>n+j}$ is a run through $\mathfrak{f}$, and for all $n' > n+j$ we have $n \sim_{\mathcal{Q}_0} n' - (j-i)$ whenever $n \sim_{\mathcal{R}} n'$. Thus we obtain a FSA-subquasimodel

$$\left\langle I_1 * I_2^{\leq 0} * I_3, \mathcal{Q}_0 \right\rangle$$

of $\langle I, \mathcal{R}\rangle$ such that there is a run $r_1 \in \mathcal{Q}_0$ through $\mathfrak{f}$ realizing $\Diamond^+ X$ in

$$m_1 \leq 2^{2\cdot|term(\varphi)|} \cdot \natural(\varphi)$$

steps and such that, for all $n' > n + m_1$ we have $n \sim_{\mathcal{Q}_0} n' - (j-i)$ whenever $n \sim_{\mathcal{R}} n'$. In particular, $[n]_{\mathcal{Q}_0}$ is infinite.

After that we consider another formula $\Diamond^+ Z \in l_\mathfrak{f}(x)$, for some $x \in W_\mathfrak{f}$ and assume that it is realized in $m_2 > m_1$ steps in r_1. Using Lemma 39 once again (and deleting states in the interval $I_3(m_1), \dots, I_3(m_2)$) we construct a FSA-subquasimodel

$$\left\langle I_1 * I_2^{\leq 0} * I_3^{\leq m_1} * I_4, \mathcal{Q}_1 \right\rangle$$

of $\langle I, \mathcal{R}\rangle$ and a run r_2 through $\mathfrak{f}$ realizing both $\Diamond^+ X$ and $\Diamond^+ Z$ in $2 \cdot 2^{2\cdot|term(\varphi)|} \cdot \natural(\varphi)$ steps, with $[n]_{\mathcal{Q}_1}$ being infinite.

Having analyzed all distinct terms of the form $\Diamond^+ X$ which occur in some $l_\mathfrak{f}(x)$ we obtain a FSA-subquasimodel

$$\left\langle I_1 * I_2^{\leq 1} * I', \mathcal{Q}' \right\rangle$$

of $\langle I, \mathcal{R}\rangle$ with finite $\mathcal{Q}'$ and a run $r' \in \mathcal{Q}'$ through $\mathfrak{f}$ realizing all $\Diamond^+$-terms in $m' \leq 3 \cdot |term(\varphi)| \cdot 2^{2\cdot|term(\varphi)|} \cdot \natural(\varphi)$ steps. The class $[n]_{\mathcal{Q}'}$ is infinite.

Then we consider in the same manner another labelled fork $\mathfrak{f}' \in \boldsymbol{F}_n$. However, this time we can delete states only after $I'(m')$. And so forth. Thus we arrive at a FSA-subquasimodel

$$\left\langle I_1 * I_2^{\leq 0} * I'', \mathcal{Q}'' \right\rangle$$

of $\langle I, \mathcal{R}\rangle$ with infinite $[n]_{\mathcal{Q}''}$ and such that all terms of the form $\Diamond^+ X$ which occur in some $l_{\boldsymbol{F}_n}(x)$ are realized by some $r \in \mathcal{Q}'$ in

$$\leq 2^{2\cdot|term(\varphi)|} \cdot 3 \cdot |term(\varphi)| \cdot 2^{2\cdot|term(\varphi)|} \cdot \natural(\varphi)$$

steps.

To comply with (iii) we need $2^{2\cdot|sub(\varphi)|} \cdot \natural(\varphi) \cdot |sub(\varphi)|$ further steps.

Finally, we need at most $\natural(\varphi)$ new states to comply with (iv).

(v) is satisfied by the construction. ❑

Definition 41 (suitable pair). Say that an ordered pair $\langle \mathfrak{f}, l_\mathfrak{f}\rangle$, $\langle \mathfrak{g}, l_\mathfrak{g}\rangle$ of labelled forks for φ is *suitable* if there is an isomorphism σ from $\mathfrak{f}$ onto $\mathfrak{g}$ such that, for all $\bigcirc t$, $\Box^+ t$, and $\Diamond^+ t$ in $term(\varphi)$ and all $x \in W_\mathfrak{f}$,

- $\bigcirc t \in l_{\mathfrak{f}}(x)$ iff $t \in l_{\mathfrak{g}}(\sigma(x))$,
- $\Box^{+} t \in l_{\mathfrak{f}}(x)$ iff $t, \Box^{+} t \in l_{\mathfrak{g}}(\sigma(x))$,
- $\Diamond^{+} t \in l_{\mathfrak{f}}(x)$ iff $t \in l_{\mathfrak{g}}(\sigma(x))$ or $\Diamond^{+} t \in l_{\mathfrak{g}}(\sigma(x))$.

An ordered pair Φ, Ψ of formula types is suitable if for every $\psi_1 \mathcal{U} \psi_2$:

$$\psi_1 \mathcal{U} \psi_2 \in \Phi \Leftrightarrow \psi_2 \in \Psi \text{ or } \psi_1 \mathcal{U} \psi_2 \in \Psi.$$

Now the decidability of the satisfiability problem for formulas in $\mathcal{ST}_2$ in FSA-models follows from the following result:

Theorem 42. *A formula φ is satisfiable in a tt-model with FSA iff there exist two state functions I_1, I_2 of length*

- $l_1 \leq \sharp(\varphi)$,
- $l_2 \leq \natural(\varphi) \cdot (2^{4 \cdot |term(\varphi)|} \cdot 3 \cdot |term(\varphi)| + 2^{2 \cdot |sub(\varphi)|} \cdot \natural(\varphi) \cdot |sub(\varphi)| + 1)$.

with

$$I = I_1 * I_2^*$$

such that the following holds (for $I(n) = \langle \boldsymbol{F}_n, \Phi_n \rangle$):

1. (a) $\varphi \in \Phi_0$,
   (b) *for $i < l_1 + l_2$, Φ_i, Φ_{i+1} is suitable,*
   (c) *$\Phi_{l_1+l_2-1}, \Phi_{l_1}$ is suitable,*
   (d) *for every $i \leq l_1$ and every $\psi \mathcal{U} \psi_2 \in \mathcal{U} \in \Phi_i$: $\psi_1 \mathcal{U} \psi_2$ is realized in $l_1 + l_2 - i$ steps;*
2. *for every $i < l_1 + l_2$ and every labelled fork $\mathfrak{h}_i$ in $\boldsymbol{F}_i$ there is a sequence $\mathfrak{h}_0, \dots, \mathfrak{h}_{l_1+l_2-1}$ such that*
   (a) *$\mathfrak{h}_j \in \boldsymbol{F}_j$, for every $j < l_1 + l_2$,*
   (b) *the pair $\mathfrak{h}_j, \mathfrak{h}_{j+1}$ is suitable, for every $j < l_1 + l_2 - 1$,*
   (c) *$\mathfrak{h}_{l_1+l_2-1}, \mathfrak{h}_{l_1}$ is suitable;*
3. *for every $i \leq l_1$ and every labelled fork $\mathfrak{h}_i$ in $\boldsymbol{F}_i$, there is a sequence $\mathfrak{h}_0, \dots, \mathfrak{h}_{l_1+l_2-1}$ such that*
   (a) *every $\Diamond^{+} t$ from $\mathfrak{h}_i$ is realized in $l_1 + l_2 - i$ steps in $\mathfrak{h}_0, \dots, \mathfrak{h}_{l_1+l_2-1}$,*
   (b) *$\mathfrak{h}_j \in \boldsymbol{F}_j$, for every $j < l_1 + l_2$,*
   (c) *the pair $\mathfrak{h}_j, \mathfrak{h}_{j+1}$ is suitable, for every $j < l_1 + l_2 - 1$,*
   (d) *$\mathfrak{h}_{l_1+l_2-1}, \mathfrak{h}_{l_1}$ is suitable.*

Proof Firstly we show that φ is satisfiable in a FSA-quasimodel whenever we find an I satisfying the conditions of the theorem. To this end suppose I satisfies those conditions. It suffices to show that there exists a set of runs $\mathcal{R}$ in I such that $\langle I, \mathcal{R} \rangle$ is a FSA-quasimodel.

Say that a sequence $\mathfrak{h}_0, \dots, \mathfrak{h}_{l_1+l_2-1}$ is of *type 1* (*type 2*) if it satisfies condition 2 (respectively, condition 3 for $i = l_1$) in the formulation of the theorem. Clearly, there are finitely many sequences of type 1, and every sequence of type 2 is also a sequence of type 1.

Let $\mathcal{R}$ consist of all infinite words of the form

$$s_1 * (s_2^{\geq l_1} * s_3^{\geq l_1})^* \text{ and } s_1 * (s_3^{\geq l_1} * s_2^{\geq l_1})^*,$$

where s_1, s_3 are sequences of type 1 and s_2 is a sequence of type 2 such that

- the pair $s_1(l_1 + l_2 - 1)$, $s_2(l_1)$ is suitable and
- $s_2(l_1) = s_3(l_1)$.

It is readily checked that every such word is a run in I and that $\langle I, \mathcal{R}\rangle$ is a FSA-quasimodel for φ.

Conversely, it follows from Lemma 40 that there exists an I satisfying the conditions above whenever φ is satisfied in a FSA-quasimodel for φ. ❑

6 Temporal models of Euclidean space

Although RCC was formulated as a first-order theory that can be interpreted in arbitrary topological spaces, of course the intended models for various applications are one-, two-, or three-dimensional Euclidean spaces, i.e., $\mathbb{R}^n$ for $n = 1, 2, 3$ with the standard interior operator.[3] Renz [27] showed that for pure RCC-8-formulas satisfiability in arbitrary topological spaces coincides with satisfiability in $\mathbb{R}$, and so in $\mathbb{R}^n$ for any $n > 0$; $\mathbb{R}^3$ is enough to realize any set of satisfiable RCC-8 formulas using only connected regions.

Let us observe first that this result of [27] cannot be generalized to RCC-8 extended with the operation $\vee$ intended to form unions of regions.

Proposition 43. *There exists a satisfiable* RCC-8 *formula φ with $\vee$ which is not satisfiable in any connected*[4] *topological space. In particular, φ is not satisfiable in $\mathbb{R}^n$ for any $n \geq 1$.*

Proof Take the conjunction φ of the following predicates:

$$\mathsf{EQ}(X_1 \cup X_2, Y), \quad \mathsf{NTPP}(X_1, Y), \quad \mathsf{NTPP}(X_2, Y), \quad \mathsf{DC}(Y, Z).$$

Clearly, φ is satisfied in the topological space consisting of three points and having the identical interior operator.

Note now that if φ holds in some topological space, then the region $X_1 \cup X_2$ is closed and included in the interior of Y. On the other hand, it coincides with Y. Hence Y is both closed and open. However, Y is not the whole space because it is disjoint with Z. ❑

Since the operation of forming unions of regions is implicitly available in the language $\mathcal{ST}_2$ (in the form of $\diamond^+$), we obtain the following

Proposition 44. *There is an $\mathcal{ST}_2$-formula satisfiable in some tt-model with FSA, but not in a model based on a connected topological space, in particular $\mathbb{R}^n$, for any $n \geq 1$.*

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[3] Cohn [10] notes, however, that in some applications discrete or even finite topological spaces may be preferable.

[4] A space is called connected if it is not the union of two disjoint non-empty open sets.

Proof Let φ be the conjunction of the predicates:

$$\mathsf{EQ}(\Diamond^+ X, Y), \quad \mathsf{NTPP}(\bigcirc X, Y), \quad \mathsf{NTPP}(\bigcirc\Diamond^+ X, Y), \quad \mathsf{DC}(Y, Z).$$

As in the proof above, we can show that if φ holds at some moment of time, then Y at that moment must be clopen. ❑

Fortunately, this is not the case for $\mathcal{ST}_1$-formulas.

Theorem 45. *If a set of $\mathcal{ST}_1$-formulas is satisfiable in a tt-model then it is also satisfiable in a model based on $\mathbb{R}^n$, for any $n \geq 1$.*

Proof As follows from the proof of Theorem 21, it suffices to show that if a set ? of RCC-8 predicates and their negations has a topological model determined by a Kripke model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ in which $\mathfrak{F}$ is a disjoint union of countably many forks, then ? has a topological model based on $\mathbb{R}$.

So let us assume that ? has such a model $\mathfrak{M}$ and $\mathfrak{F}$ is the disjoint union of ω forks $\mathfrak{f}_n$, $n < \omega$, with three points: a_n, the root, and its two successors b_n and c_n.

Denote by $\mathcal{X}^n_{ab}$ the set of all region variables X such that

$$\mathfrak{V}(X) \cap \mathfrak{f}_n = \{a_n, b_n\}.$$

Analogously, $\mathcal{X}^n_{ac}$, and $\mathcal{X}^n_{abc}$ are the sets of all region variables X such that

$$\mathfrak{V}(X) \cap \mathfrak{f}_n = \{a_n, c_n\},$$
$$\mathfrak{V}(X) \cap \mathfrak{f}_n = \{a_n, b_n, c_n\},$$

respectively. And let $\mathcal{X}^n = \mathcal{X}^n_{ab} \cup \mathcal{X}^n_{ac} \cup \mathcal{X}^n_{abc}$.

Define a partial order $\preceq$ on $\mathcal{X}^n$ by taking

$$X \preceq Y \quad \text{iff} \quad \mathfrak{M} \models \forall (X \to Y).$$

For each $n < \omega$, we then choose three maps

- $f^n_{ab} : \mathcal{X}^n_{ab} \mapsto (0, 0.2)$,
- $f^n_{ac} : \mathcal{X}^n_{ac} \mapsto (0, 0.2)$,
- $f^n_{abc} : \mathcal{X}_{abc} \mapsto (0.3, 0.4)$

in such a way that, for every $\bar{e} \in \{ab, ac, abc\}$ and all $X, Y \in \mathcal{X}^n_{\bar{e}}$,

$$f^n_{\bar{e}}(X) < f^n_{\bar{e}}(Y) \quad \text{if} \quad X \preceq Y,$$

and

$$f^n_{\bar{e}}(X) = f^n_{\bar{e}}(Y) \quad \text{if} \quad X \preceq Y \,\&\, Y \preceq X.$$

Clearly, such maps exist.

Then we put

- $\mathfrak{a}^n(X) = [n - f^n_{ab}(X), n]$ for every $X \in \mathcal{X}^n_{ab}$,
- $\mathfrak{a}^n(X) = [n, n + f^n_{ac}(X)]$ for every $X \in \mathcal{X}^n_{ac}$,
- $\mathfrak{a}^n(X) = [n - f^n_{abc}(X), n + f^n_{abc}(X)]$ for every $X \in \mathcal{X}^n_{abc}$,

- $\mathfrak{a}^n(X) = \emptyset$ for every $X \notin \mathcal{X}^n$.

Finally, let

$$\mathfrak{a}(X) = \bigcup_{n<\omega} \mathfrak{a}^n(X)$$

for all region variables X.

It is a matter of routine now to show that ? holds in the topological space $\mathbb{R}$ under the assignment $\mathfrak{a}$. We will consider here only two cases.

Suppose $\mathsf{EC}(X,Y) \in ?$. Then there is $n < \omega$ such that

$$a_n \in \mathfrak{V}(X) \cap \mathfrak{V}(Y), \tag{2}$$

but $\bigcup_{n<\omega}\{b_n, c_n\}$ is disjoint with $\mathfrak{V}(X) \cap \mathfrak{V}(Y)$. It follows that $\mathfrak{a}(X)$ and $\mathfrak{a}(Y)$ are externally connected by all n for which (2) holds.

Suppose $\mathsf{NTPP}(X,Y) \in ?$. Then $\mathfrak{a}(X) \subseteq \mathfrak{a}(Y)$ and $\mathfrak{a}(Y) - \mathfrak{a}(X) \neq \emptyset$. It remains to show that $\mathfrak{a}(X)$ is included in the interior of $\mathfrak{a}(Y)$. Let $x \in \mathfrak{a}(X)$. Then $x \in \mathfrak{a}^n(X)$, for some $n < \omega$. It suffices to show that r is in the interior of $\mathfrak{a}^n(Y)$. Three cases are possible:

Case 1: $X \in \mathcal{X}^n_{ab}$. Then $Y \in \mathcal{X}^n_{ab}$ or $Y \in \mathcal{X}^n_{abc}$. If $Y \in \mathcal{X}^n_{ab}$, then x is in the interior of $\mathfrak{a}^n(X)$, since $f^n_{ab}(X) < f^n_{ab}(Y)$ (recall that $\mathfrak{M} \models \forall(X \to Y)$ but not vice versa). If $Y \in \mathcal{X}^n_{abc}$, then x is in the interior of $\mathfrak{a}^n(Y)$ since the range of f^n_{ab} is contained in $(0, 0.2)$ while the range of f^n_{abc} is contained in $(0.3, 0.4)$.

Case 2: $X \in \mathcal{X}^n_{ac}$. This case is dual to Case 1.

Case 3: $X \in \mathcal{X}^n_{abc}$. Then $Y \in \mathcal{X}^n_{abc}$ (since $\mathfrak{M} \models \forall(X \to Y)$) and $f^n_{abc}(X) < f^n_{abc}(Y)$ from which we deduce that x is in the interior of $\mathfrak{a}^n(Y)$. ❑

7 Conclusion

In this paper we constructed a family of logics intended for qualitative knowledge representation and reasoning about the behaviour of spatial regions in time. We proved that reasoning in these logics is effective and estimated its computational complexity. We also found out the relationship between topological temporal models based on abstract topological spaces and those on Euclidean ones.

The obtained results make the first step in the study of effective spatio-temporal formalisms. Many interesting problems remain open for investigation. Here are some of them, connected with the logics constructed in the paper.

(1) Theorems 4, 7, and 9 establish only upper bounds for the complexity of the satisfiability problem for $\mathcal{ST}^+_i$, $i > 0$. Do they coincide with the lower bounds?

(2) It may be of interest to extend the logics $\mathcal{ST}_i$ with infinitary operators which allow us to construct formulas of the form $\bigwedge_{n\in\mathbb{N}} R(X, \bigcirc^n Y)$ and $\bigvee_{n\in\mathbb{N}} R(X, \bigcirc^n Y)$ to say, for instance, that $\bigvee_{n\in\mathbb{N}} \mathsf{P}(Russia, \bigcirc^n EU)$, i.e., some time in the future the whole Russia as it is today will become part of the EU.

(3) FSA assumes that we can apply $\Box^+$ and $\Diamond^+$ only to region terms that have finitely many possible states. To allow for infinite sets of states, one may consider models that are 'compact' in the sense that regions can change infinitely often and have infinitely many possible states, but there are finitely many maximal and minimal (with respect to $\subseteq$) states starting from each $n \in \mathbb{N}$. We

conjecture that the satisfiability problem for $\mathcal{ST}_i^+$-formulas in compact topological temporal models is decidable.

(4) Theorem 15 realises region variables as arbitrary regular closed sets of $\mathbb{R}^n$. What if we require those sets to be connected? (See e.g. [27].)

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